TRANSACTIONS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XXIV. PART III.—FOR THE SESSION 1866–67.

CONTENTS.

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LAWS

OF THE

ROYAL SOCIETY OF EDINBURGH,

AS REVISED 31st OCTOBER 1866.

LAWS.

[By the Charter of the Society (printed in the *Transactions*, Vol. VI. p. 5), the Laws cannot be altered, except at a Meeting held one month after that at which the Motion for alteration shall have been proposed.]

I.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows. Title.

II.

Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter.* The fees of Ordinary Fellows residing in Scotland.

III.

All Fellows who shall have paid Twenty-five years' annual contribution shall be exempted from farther payment. Payment to cease after 25 years.

IV.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt from any farther payment. In the case of any Resident Fellow ceasing to reside in Scot- Fees of Non-Resident Ordinary Fellows. Case of Fellows becoming Non-Resident.

* At the Meeting of the Society, on the 5th January 1857, when the reduction of the Contributions from £3, 3s., to £2, 2s., from the 11th to the 25th year of membership, was adopted, it was resolved that the existing Members shall share in this reduction, so far as regards their future Annual Contributions.

A modification of this rule, in certain cases, was agreed to 3d January 1831.

land, and wishing to continue a Fellow of the Society, it shall be in the power of the Council to determine on what terms, in the circumstances of each case, the privilege of remaining a Fellow of the Society shall be continued to such Fellow while out of Scotland.

V.

Defaulters.

Members failing to pay their contributions for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

Privileges of Ordinary Fellows.

None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

VII.

Numbers Unlimited.

The number of Ordinary Fellows shall be unlimited.

VIII.

Fellows entitled to Transactions.

The Ordinary Fellows, upon producing an order from the TREASURER, shall be entitled to receive from the Publisher, gratis, the Parts of the Society's Transactions which shall be published subsequent to their admission.

IX.

Mode of Recommending Ordinary Fellows.

No person shall be proposed as an Ordinary Fellow without a recommendation subscribed by *One* Ordinary Fellow, to the purport below.* This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall afterwards be printed in the circulars for three Ordinary Meetings of the Society, previous to the day of the election, and shall lie upon the table during that time.

X.

Honorary Fellows, British and Foreign.

Honorary Fellows shall not be subject to any contribution. This class shall

* " A. B., a gentleman well skilled in several branches of Science (*or Polite Literature, as the case* " *may be*), being to my knowledge desirous of becoming a Fellow of the Royal Society of Edin- " burgh, I hereby recommend him as deserving of that honour, and as likely to prove a useful and " valuable Member."

This recommendation to be accompanied by a request of admission signed by the Candidate.

consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom Twenty may be British subjects, and Thirty-six may be subjects of foreign states.

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X. Royal Personages.

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below*) subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one meeting, and printed in the circular for the meeting at which the Ballot is to take place. Recommendation of Honorary Fellows. Mode of Election.

XIII.

The election of Ordinary Fellows shall take place at the Ordinary Meetings of the Society. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote. Election of Ordinary Fellows.

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of every month from November to June inclusive. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make. Ordinary Meetings.

XV.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall The Transactions.

* We hereby recommend ____________________
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

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To the President and Council of Royal Society of Edinburgh.

deem it expedient to publish in the *Transactions* of the Society, and shall superintend the printing of the same.

XVI.

How Published.

The Transactions shall be published in Parts or *Fasciculi* at the close of each Session, and the expense shall be defrayed by the Society.

The Council.

There shall be elected annually, for conducting the publications and regulating the private business of the Society, a Council, consisting of a President; Six Vice-Presidents, two at least of whom shall be resident; Twelve Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

XVII.

Retiring Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XVIII.

Election of Office-Bearers.

An Extraordinary Meeting for the Election of Office-Bearers shall be held on the fourth Monday of November annually.

XIX.

Special Meetings; how called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XX.

Treasurer's Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the Extraordinary Meeting in November, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXI.

Auditor.

At the Extraordinary Meeting in November, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intromissions.

XXII.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes, he shall, when necessary, employ a clerk, to be paid by the Society. General Secretary's Duties.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the donors' names, to the Curator of the Library and Museum: they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty. Secretaries to Ordinary Meetings.

XXIII.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Fellows. Curator of Museum and Library.

XXIV.

All Articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations, as the Council from time to time shall appoint. Use of Museum and Library.

XXV.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date. Register Book.

ROYAL SOCIETY OF EDINBURGH.

THE KEITH, BRISBANE, AND NEILL PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The Keith Prize, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1867–68, for the "best communication on a scientific subject, communicated, in the first instance, to the Royal Society during the Sessions 1865–66 and 1866–67." Preference will be given to a paper containing a discovery.

II. MAKDOUGALL BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper writen on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1868–69, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 1st June 1868.

3. The competition is open to all men ofscience.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society. They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the Paper shall be published in the Transactions.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented to the Society during the Sessions 1866–67 and 1867–68, whether they may have been given in with a view to the Prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate,

1. The NEILL PRIZE, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1868–69.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented to the Society during the three years preceding the 1st May 1868,—or failing presentation of a Paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

AWARDS OF THE KEITH, MAKDOUGALL BRISBANE, AND NEILL PRIZES, SINCE 1859.

AWARD OF THE KEITH PRIZE.

19TH BIENNIAL PERIOD, 1863–65. Principal FORBES, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

AWARD OF THE MAKDOUGALL BRISBANE PRIZE

4TH BIENNIAL PERIOD, 1864–66. Not yet awarded.

AWARD OF THE NEILL PRIZE.

3D TRIENNIAL PERIOD, 1862–65. ANDREW CROMBIE RAMSAY, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

NON-RESIDENT MEMBER,

ELECTED UNDER THE OLD LAWS.

Sir Richard Griffiths, Bart., Dublin.

LIST OF HONORARY FELLOWS.

His Majesty the King of the Belgians.
His Royal Highness the Prince of Wales.

FOREIGNERS (LIMITED TO THIRTY-SIX.)

Louis Agassiz,	*Cambridge, Massachusetts.*
Alexander Dallas Bache,	*Washington.*
J. B. A. L. Léonce Elie de Beaumont,	*Paris.*
Robert Wilhelm Bunsen,	*Heidleberg.*
Victor Cousin,	*Paris.*
James D. Dana, LL.D.,	*Newhaven, Connecticut.*
Jean Baptiste Dumas,	*Paris.*
Charles Dupin,	*Do.*
Christien Gottfried Ehrenberg,	*Berlin.*
Johann Franz Encke,	*Do.*
Pierre Marie Jean Flourens,	*Paris.*
Jean Bernard Leon Foucault,	*Do.*
Elias Fries,	*Upsala.*
François Pierre Guillaume Guizot,	*Paris.*
Wilhelm Karl Haidinger,	*Vienna.*
Christopher Hansteen,	*Christiania.*
Hermann Helmholtz,	*Heidelberg.*
Albert Kölliker,	*Wurzburg.*
J. Lamont,	*Munich.*
Richard Lepsius,	*Berlin.*
Rudolph Leuckart,	*Giessen.*
Urbain Jean Joseph Leverrier,	*Paris.*
Baron Justus von Liebig,	*Munich.*

Carl Friedrich Philip von Martius,	*Munich.*
Henry Milne-Edwards,	*Paris.*
Theodore Mommsen,	*Berlin.*
Adolphe Pictet,	*Geneva.*
Lambert Adolphe Jacques Quetelet,	*Brussels.*
Henri Victor Regnault,	*Paris.*
Auguste De la Rive,	*Geneva.*
Gustav Rose,	*Berlin.*
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Angelo Secchi,	*Rome.*
Karl Theodore von Siebold,	*Munich.*
Bernard Studer,	*Berne.*

BRITISH SUBJECTS (LIMITED TO TWENTY, BY LAW X.)

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Arthur Cayley, Esq.,	*Cambridge.*
Charles Darwin, Esq.,	*Down, Bromley, Kent.*
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Thomas Graham, Esq.,	*Do.*
Sir John Frederick William Herschel, Bart.,	*Collingwood.*
William Lassell, Esq.,	*Liverpool.*
Rev. Dr Humphrey Lloyd,	*Dublin.*
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John Stuart Mill,	*Do.*
Sir Roderick Impey Murchison,	*Do.*
Richard Owen, Esq.,	*Do.*
Earl of Rosse,	*Parsonstown.*
Lieut.-General Edward Sabine, R.A.	*London.*
George Gabriel Stokes,	*Cambridge.*
William Henry Fox Talbot, Esq.,	*Lacock Abbey, Wiltshire.*
Alfred Tennyson,	*Freshwater, Isle of Wight.*

The following Public Institutions and Individuals are entitled to receive Copies of the Transactions and Proceedings of the Royal Society of Edinburgh:—

ENGLAND.

The British Museum.
The Bodleian Library, Oxford.
The University Library, Cambridge.

The Royal Society.
The Linnean Society.
The Society for the Encouragement of Arts.
The Geological Society.
The Royal Astronomical Society.
The Royal Asiatic Society.
The Zoological Society.
The Royal Society of Literature.
The Royal Horticultural Society.
The Royal Institution.
The Royal Geographical Society.
The Statistical Society.
The Institution of Civil Engineers.
The Institute of British Architects.
The Hydrographical Office, Admiralty.
The Medico-Chirurgical Society.
The Athenæum Club.
The Cambridge Philosophical Society.
The Manchester Literary and Philosophical Society.
The Yorkshire Philosophical Society.
The Chemical Society of London.
The Museum of Economic Geology.
The United Service Institution.
The Royal Observatory, Greenwich.
The Leeds Philosophical and Literary Society.
The Historic Society of Lancashire and Cheshire.
The Royal College of Surgeons of England.

SCOTLAND.

Edinburgh, University Library.
... Advocates' Library.
... College of Physicians.
Edinburgh, Highland and Agricultural Society.
... Royal Medical Society,
... Royal Physical Society.
... Royal Scottish Society of Arts.
Glasgow, University Library.
St Andrews, University Library.
Aberdeen, University Library.

IRELAND.

The Library of Trinity College, Dublin.
The Royal Irish Academy.

COLONIES, &c.

The Asiatic Society of Calcutta.
Library of Geological Survey, Calcutta.
The Literary and Historical Society of Toronto.
University of Sydney.

CONTINENT OF EUROPE.

Amsterdam, Royal Institute of Holland.
Berlin, Royal Academy of Sciences.
... Physical Society.
Berne, Society of Swiss Naturalists.
Bologna, Academy of Sciences.
Bonn, Cæsarean Academy of Naturalists.
Brussels, Royal Academy of Sciences.
Buda, Literary Society of Hungary.
Copenhagen, Royal Academy of Sciences.
Frankfort, the Senkenbergian Museum.
Geneva, Natural History Society.
Giessen, University Library.
Göttingen, University Library.
Haarlem, Natural History Society.
Leipzig, Royal Saxon Academy.
Lille, Royal Society of Sciences.
Lisbon, Royal Academy of Sciences.
Lyons, Agricultural Society.
Milan, Royal Institute.

Moscow, Imperial Academy of Naturalists.
Munich, Royal Academy of Sciences of Bavaria (2 copies).
Neufchatel, Museum of Natural History.
Paris, Royal Academy of Sciences.
... Geographical Society.
... Royal Society of Agriculture.
... Society for Encouragement of Industry.
... Geological Society of France.
... Ecole des Mines.
... Marine Depôt.
... Museum of Jardin des Plantes.
Rotterdam, Batavian Society of Experimental Philosophy.
Stockholm, Royal Academy of Sciences.
St Petersburgh, Imperial Academy of Sciences.
... Pulkowa Observatory.
Turin, Royal Academy of Sciences.
... M. Michelotti.
Upsala, Society of Sciences.
Venice, Royal Institute.
Vienna, Imperial Academy of Sciences.
... Geological Society.
... Geologico-Botanical Society.

UNITED STATES OF AMERICA.

Boston, the Bowditch Library.
... Academy of Arts and Sciences.
New York, State Library.
Philadelphia, American Philosophical Society.
... Academy of Natural Sciences.
Washington, the Smithsonian Institution.

(*All the Honorary and Ordinary Fellows of the Society are entitled to the Transactions and Proceedings.*)

The following Institutions and Individuals receive the Proceedings only:—

ENGLAND.

The Scarborough Philosophical Society.
The Whitby Philosophical Society.
The Newcastle Philosophical Society.
The Geological Society of Cornwall.
The Ashmolean Society of Oxford.
The Literary and Philosophical Society of Liverpool.

SCOTLAND.

The Philosophical Society of Glasgow.
The Botanical Society of Edinburgh.
The Geological Society of Edinburgh.

IRELAND.

The Natural History Society of Dublin.

COLONIES.

The Literary and Philosophical Society of Quebec.
The Library of the Geological Survey, Canada.
The Literary Society of Madras.
China Branch of Asiatic Society, Hongkong.
North China Branch of the Royal Asiatic Society, Shanghae.

CONTINENT OF EUROPE.

Utrecht, the Literary and Philosophical Society.
Paris, Editor of L'Institut.
Cherbourg, Society of Natural Sciences.
Catania in Sicily, Academia Govenia de Scienze Naturali.

UNITED STATES.

H. T. Parker, Esq., Harvard College, Cambridge.

LIST OF THE ORDINARY FELLOWS OF THE SOCIETY.

N.B.—*Those marked * are Annual Contributors.*

1846 *Alex. J. Adie, Esq., Rockville, Linlithgow
1866 *Col. Sir James E. Alexander of Westerton
1830 Sir Archibald Alison, Bart., Possil House, Lanark
1848 Dr James Allan, Inspector of Hospitals, Portsmouth
1856 *Dr G. J. Allman (SECRETARY), Professor of Natural History, 21 Manor Place
1849 *David Anderson, Esq., Moredun, Edinburgh
1845 *Dr Thomas Anderson, Prof. Chemistry, Univ., Glasgow
1823 Warren Hastings Anderson, Esq., Isle of Wight
1840 James Anstruther, Esq., W.S.
1862 *T. C. Archer, Esq., Director of the Industrial Museum, 9 Argyll Square 10
1849 *His Grace the Duke of Argyll, Inverary Castle
1822 Dr G. Walker Arnott, Prof. Botany, Univ., Glasgow
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1835 Dr J. H. Balfour (GENERAL SECRETARY), Professor of Medicine and Botany, 27 Inverleith Row
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1830 Dr Thomas Barnes, Carlisle
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1844 *Dr Begbie, 10 Charlotte Square
1843 *Dr Bennett, Professor of Institutes of Medicine, 1 Glenfinlas Street 20
1861 *George Berry, Esq., Rosefield Cottage, Portobello
1866 *Adam Black, Esq., 38 Drummond Place
1857 *Dr James Black, 2 George Square
1850 *Hugh Blackburn, Esq., Prof. Mathematics, Univ., Glasg.
1863 *Professor Blackie, 24 Hill Street
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1822 William Bonar, Esq., East Warriston
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1856 *Thomas Cleghorn, Esq., Advocate, 26 Queen Street
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1842 Dr John Davy, Lesketh How, Ambleside
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1824 Dr Robert Hamilton, 11 North Merchiston Place
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1865 *Charles Lawson, jun., Esq., 34 George Square
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1858 *James Leslie, Esq., C.E., 2 Charlotte Square
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1864 *William Lindsay, Esq., Hermitage-Hill House, Leith
1857 Thomas Login, Esq., C.E., Pegu
1861 *Professor Lorimer, Advocate, 21 Hill Street
1849 *Dr W. H. Lowe, Balgreen, Slateford 140
1855 *Dr Stevenson Macadam, 25 Brighton Place, Portobello
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1866 *John M'Culloch, Esq., Banker, 11 Duke Street
1820 Dr Wm. Macdonald, Prof. Civ. and Nat. Hist., St Andrews
1847 *W. Macdonald Macdonald, Esq., St Martins
1860 *Professor MacDougall, 9 Buckingham Terrace
1861 *Dr A. E. Mackay, R.N.
1840 John Mackenzie, Esq., 11 Abercromby Place
1843 *Dr A. Douglas Maclagan (Curator), Prof. of Medical Jurisprudence, 28 Heriot Row
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1855 Dr Wright, Cheltenham
1864 *Robert S. Wyld, Esq., W.S., 19 Inverleith Row
1861 *James Young, Esq., Limefield, Mid-Calder
1863 *Dr John Young, Professor of Natural History, Glasgow 277

Fellows elected between the commencement of the Session and the 1st January of the following year are entered under the latter date, by which their Subscriptions are regulated:—Thus, Fellows elected in December 1865 have the date of 1866 prefixed to their names.

XXXIII.—*On the Colours of the Soap-Bubble.* By Sir David Brewster, K.H., F.R.S. (Plate XXXIII.)

(Read 21st January 1867.)

The phenomena and colours of the soap-bubble have been the subject of more frequent observation than any other facts in science. For nearly two centuries they have afforded amusement and instruction to the young, and have exercised the genius of the most distinguished philosophers. Hook* and Newton† ascribed the beautiful colours of the soap-bubble to the different degrees of thickness of the attenuated film, and this opinion has been implicitly adopted by every optical writer down to the present day. In the latest and best Treatise on Light, Sir John Herschel‡ expresses the current theory more distinctly than others, when he says, "that the brilliant colours which appear on soap-bubbles consist of a regular succession of hues disposed in the same order, and determined, obviously not by any colour in the medium itself, in which they are formed, or on whose surfaces they appear, but solely by its greater or less thickness. . . . . It is at first uniformly white, but as it grows thinner and thinner, by the subsidence of its particles, colours appear to begin at its top, where thinnest, which grow more and more vivid, and arrange themselves in beautiful horizontal zones about the highest point, as a centre."

Of the correctness of this theory I never entertained a doubt, till I resumed the study of the subject, while repeating the beautiful experiments of Professor Plateau, "On the Figures of equilibrium of a liquid mass without gravity." The colours exhibited by plane, convex, and concave liquid films thus came under my notice, and led me to the true cause of the colours of the soap-bubble.

Dr Thomas Young§ is the only author, so far as I know, that has described these colours, as produced by a film stretched across the mouth of a wine-glass, and it is strange that he did not observe certain changes in the coloured bands produced by the simple motion of the glass, which might have led him to their true explanation. He contents himself with giving an incorrect coloured drawing of the bands in their first stage, and adopts the usual theory of their formation.‖

* Birch's Hist. of the Royal Society, vol. iii. p. 29.

† Optics, 3d edition, p. 187.

‡ Treatise on Light, § 633, p. 462.

§ Elements of Natural Philosophy, vol. i. p. 469, and plate xxx. fig. 448. The dark band in this figure does not exist.

‖ "When a film of soapy water," says Dr Young, "is stretched over a wine-glass, and placed in a vertical position, its upper edge becomes extremely thin, and apparently black, while the parts below are divided by horizontal lines into a series of coloured bands."—*Id.* p. 458.

As the colours of the soap-bubble cannot be well studied on the bubble itself even when defended from the action of the air by "a clear glass," as NEWTON did, I have employed plane, convex, and concave films of all sizes, up to 3¼ inches in diameter, as obtained upon glasses, conical or cylindrical, or upon tubes of glass or metal closed at one end, or open at both. I have also employed wires of different metals bent into various shapes, rectangular, triangular, and curvilinear, but in the following experiments I have principally made use of a cylindrical wine-glass, which gives a film 1¼ inch in diameter, and unless otherwise stated, I have obtained the films from the common solution of soap and water, sufficiently strong to give an ordinary bubble.

I. *Phenomena of Colour in a Vertical Plane Film.*

1. When the film has been newly formed and held vertically for a second or two, it exhibits at its apex *six*, *seven*, or *eight* horizontal bands of colour, the uppermost white and orange-red, with a little black above them, being the first order of NEWTON'S scale. In a few seconds the different orders increase in width, descending, and becoming in succession, as shown in Figs. 1–6, eight, seven, six, four, three, &c., in number, the film when unprotected generally bursting when four or five bands cover the whole of it, as shown in Figs. 4 and 5. Previous to bursting, the black portion of the first order of colours has appeared at the apex, as shown in Fig. 3.

In order to observe the changes in the state of the bands, after their number is reduced to five or four, we must protect the film by a watch-glass, or a piece of plate glass. When this is done, the eighth, seventh, and sixth orders of colours gradually disappear, being succeeded by the fifth, fourth, third, and second orders, which, along with the first, cover the film. After a while the third and second orders disappear, and the film is covered, as in Fig. 7, with the *black* and *white* of the first order. In a few minutes the *white* of the first order disappears, and the black band covers the whole film. Before the *black* film is complete, it often advances with an irregular margin, and throws out filaments into the white band, as shown in Figs. 5 and 6. It very frequently includes also minute systems of rings of different colours, and portions of the *white* of the first order of all shapes, having within them, in constant motion, small portions of the black matter. These *white* portions gradually disappear, leaving behind them the most beautiful silvery dendritic forms, which move about till the film bursts.

When the black band is formed from a disturbed condition of the coloured bands, where a great number of separate black portions are slowly united, there may be seen a portion of the black space much darker than the rest, with a beautiful margin of white spots, and accompanied with one or two circular spaces of equal blackness, and surrounded also with white spots, so small as to require a lens to see them. This phenomenon is shown in Fig. 7. The deep black colour-

ing matter sometimes occurs in small spots and in dendritic forms, moving over a lighter black portion.

The development of the black band is very remarkable. At first, a slightly dark shade appears at the apex of the film, increasing in darkness till a patch of deep black rushes in with a brilliant margin of white silvery spots, which adhere to it even when its lower edge becomes a straight line. These silvery spots, in a very minute state, and obviously of the first order, sometimes form a sort of network within the black space, and at other times dilute, as it were, the black with white specks, seen only with the microscope.

2. If, when the film has four or five bands upon it, as in Fig. 4, or indeed any number, we make the glass turn round its axis as quickly as possible, the bands will remain horizontal, as if under the influence of gravity, proving that they are not produced by the subsidence and consequent thinning of the film, for which there is not time, as the change of place in the film is almost instantaneous.

In confirmation of this result, place the film in a horizontal position. The bands, now bent or broken, will float in irregular shapes on its surface, the film itself, which is a fixture, adhering by capillary attraction to the margin of the wine-glass.

These phenomena are seen with interesting variations, when the film is at first placed at different angles with the horizon. When it is nearly horizontal, the changes take place slowly, and the movements of the separate portions of colouring matter, to which I have given the name of *tadpoles*, is very curious, the whole film being sometimes covered by them without the trace of a band. The black matter is produced very irregularly, and in detached portions, which ultimately unite, chasing away all the other colours at the lowest part of the film. While these changes are going on, there is a considerable deposition of aqueous vapour on the lower surface of the watch-glass which covers the film.

3. When the first, second, and third, &c., bands are floating about, like oil upon water, blow upon them with the mouth gently, till they are broken down, and thus cover the film with any one colour, suppose *green* of the third order. If the film is now placed vertically, the colouring matter will resume the form of bands, as in Fig. 8, the colour produced by blowing, namely, the *green* of the third order, taking its proper place on the film in a very broad band, having bands of inferior orders above it, and very narrow bands of higher orders below it.

The phenomena produced by this process are innumerable and highly beautiful, varying with the order of the number of tints originally on the film, and with the order of the tint produced by blowing, whether the blowing commences before any bands appear, or after most of them have quitted the film.

The special colour produced over the whole film by blowing is not composed wholly of colouring matter of one order, though that matter predominates. When examined by the microscope the film exhibits a most beautiful combination of

tints, the colouring matter of various orders having collected themselves into innumerable small circular and irregularly shaped systems of rings, floating on the coloured film. Every successive blast upon the film in this state produces a new general tint, and new microscopic systems of rings.

All the preceding phenomena take place with convex and concave films, and may be observed in films one-fourth and one-half of an inch in diameter, produced upon test tubes.

II. *On the Production of Revolving Systems of Coloured Rings on the Soap-Film.*

If we place a coloured film horizontally when it is in any of the states described in the preceding experiments, and, through a narrow tube, one-eighth or one-tenth of an inch aperture, blow upon its surface in the direction of a diameter AB, Fig. 9, there will be produced two systems of rings, C, D, revolving rapidly round their centres, the system C revolving from *right* to *left*, and the system D from *left* to *right*. If we now blow in the direction AB, Fig. 10, there will be only one circular system revolving from *right* to *left*, and covering the whole surface of the film. If the film is square, or of any other form, the rings, when fully developed, will take the same shape. When there are two systems of rings, as in Fig. 9, the colours upon the rectilineal current AB are very remarkable.

These rings are produced whatever be the colours on the film, and however irregularly they may be distributed; and, in general, the two systems will be of different forms and of different colours. If, previous to blowing, however, the colours are in regular bands or in concentric circles, as they may be in films of all forms and in all positions, and if the line of the blast bisects them so that the same colours are on each side of it, then the revolving systems will be similar in form and colour.

The order of the colours in these systems is very curious. The colours of the first order, or rather the first colours of the first order, occupy the centre of the system, if they are upon the film, the *black* being in the centre, and the *white* next to it, and the successive orders in successive rings, the breadth of each ring being proportional to the quantity of colouring matter put in motion. The *black*, for example, is often a small central spot, and when that colour occupies one-half of the film, and the *white* the other half, which they often do, as in Fig. 7, the revolving portions may be wholly *black* on one side of the blast AB, and wholly *white* on the other. If there is no colour of the first order on the film previous to blowing, the first colour of the second order will occupy the centre, the largest particles of the colouring matter being carried by the centrifugal force to the outer rings of the system. If we continue the blast, the rings will gradually disappear, the colouring matter which formed them having been restored to its colourless state, and recombined with the original film. When

this is effected, and the film placed in any position, the colour bands will be again formed exactly as they would have been in a fresh film similarly placed. The revolving rings may then be again produced, and the colouring matter again combined with the soapy film.

The colours which compose the two systems of rings may be exhibited by holding the film vertically, when the colouring matter will arrange itself in bands of the different orders to which it belongs. The bands thus exhibited are very beautiful, and of great variety.

The revolving systems of rings may be produced when the film is in any other position.

III. *On the Form and Movements of the Bands and Rings in Convex and Concave Films.*

1. When convex and concave films are held vertically, the bands are formed parallel to the horizon, exactly as in plane films, and the same phenomena take place upon turning the glass round its axis.

2. When a *convex* film is placed in a true horizontal position, and defended from currents of air, the colours begin to form at the apex or summit of the film; first, a faint *green* or *red* of a high order, followed by coloured rings of inferior orders, the first coloured rings descending till the film is covered with a regular system, having black in its centre, and *red* and *green* rings occupying the circumference of the film,* as in Fig. 11.

This regular circular system may be produced more rapidly by first placing the convex film vertically, and then, when the tints of several of the first orders are developed, turning it cautiously into a horizontal position. The horizontal bands will thus be converted, in a few seconds, into a regular concentric system. This change is most easily effected with small and very convex films at the mouth of closed tubes, about half-an-inch or three-quarters in diameter, in which case we can increase the convexity even to a hemisphere by heating the tube with the hand or otherwise.

3. When a *concave*† film is placed in a true horizontal position, and defended from the air, the rings commence at the circumference of the film, and gradually extend towards the centre, when they appear as in Fig. 12, in which there are three orders of colours, from black of the first order to the red of the third order, and within this is a lens having often a number of hardly visible rings within it. This central portion of the colouring matter is obviously lenticular, as the di-

* The order of these colours, as produced upon the upper hemisphere of the soap-bubble, is described by Sir ISAAC NEWTON in his *Optics*, p. 188.

† It is difficult to obtain a good *concave* film by dipping the cylindrical wine-glass into the soap solution. During the experiments of a whole day I never failed to obtain one, but with the same glass and similar solutions I cannot now produce one. A certain mode of producing them will be found in the following paper.

minished image of a candle may be distinctly seen by reflection from its outer and inner surfaces. It is sometimes small, and sometimes the size of a sixpence, in which case it is not visibly lenticular. It sometimes increases by the appropriation of all or part of the colouring matter around it, and sometimes diminishes by its conversion into colouring matter, or partially recombining with the film.

When the circular system of rings, with the lens in its centre, is perfect, it decays in the following manner, as observed in three different films. When there is a *green* ring of the second order round the lens, eight or ten spots of *red* begin to form upon the *green*, as in Fig. 13, and sometimes shoot out into as many red spokes, like those of a wheel, joining the next red ring. The system of rings is now splendid, varying in size and colour. Outside of the spotted ring there are generally two orders of colours. The *black* is gradually enlarging, and the *white*, with its enclosed rings, diminishing. The eight or ten spots are now *green* upon a pale *greenish yellow* ground. In a minute or so they are *purple* upon a *green* ground, then *reddish*, with *white* in their centre, and moving about deforming the rings of the second order of colours. The *white* ring has now become *yellow*, the spots, with tails like tadpoles, and *green* inside, encroaching upon the second order of colours. The spots are now small, *brilliant red*, and one or two *blue* upon a *yellowish* ground. The *blue* of the first order is now being covered with *white* spots, and the *black*, with a sharp edge, constantly encroaching on the *white* and other colours. The *white* is now covered with *orange* tadpole spots, and the other spots changing their form, colour, and position. The film now burst.

In another concave film, which was very thin, there were six orders of colours, but no lens. The fifth and sixth orders disappeared by the advance of the others, and the third and fourth, occupying the centre, were broken up by numerous tadpole spots, some of them enclosing different tints. These tadpole spots are collecting, as if into a lens or large grey spot full of rings, surrounded by a *green* ring, round which is a *red* one. The film now burst from an accident.

In a third *concave* film a lens was beautifully formed within the third order of colours, and surrounded with a bright *red* ring of the third order, and ten bright *green* spots on a yellowish ground. The spots varied in colour and number till the film burst.

In a fourth concave film the changes continued till the whole film was *black*, the lens floating in the centre, and moving about with the motion of the glass.

When the concave film is thick, or the colourless fluid copious, the lens is sometimes the size of a shilling, but flat, with a few close and almost invisible rings at its margin. It then diminishes to the size of a sixpence, and takes longer time to pass throughthe changes above described.

IV. *Phenomena produced by different Solutions.*

When the films are produced from solutions of soap and water, they burst much sooner than those from glycerine solutions, and the changes in the coloured bands take place more quickly.

In a solution made by Mr DEWAR of the University Laboratory, according to PLATEAU'S receipt, with dry Oleate of Soda, the bands, in a vertical film, were produced more slowly, and the film lasted longer than when formed from the soap solution, neither of them being defended from currents of air. After standing half an hour, the third and fourth orders, and sometimes the second, were broken up by coloured tadpole portions streaming up from the lowest point of the film, as shown in Fig. 14.

With a solution which I made according to PLATEAU'S receipt, but with humid Oleate of Soda, similar phenomena were more quickly produced. The second, third, and fourth orders were more easily and completely broken up, and the film burst sooner.

When the colours upon the film, from either of these solutions, were scattered by blowing, they reformed distinctly, and after being a second and a third time scattered by blowing, they recovered their original position and distinctness.

In a glycerine solution made by a London chemist, the phenomena were entirely different from those produced by other solutions. It had become so thick and ropy that it could not be poured out of the bottle. When a film of the usual size was produced from it, and placed vertically, it exhibited none of the phenomena we have described. It gave no bands, and when convex or concave, and placed horizontally, no rings or portions of rings were produced. The colouring matter, too thick to give colours, floated on the surface of the film in currents or little streams, and sometimes remained at rest in irregular patches. From these currents or fixed portions, streams of various and brilliant colours rose to the apex of the vertical film, with different velocities, from the bottom and sides of it, jostling one another, and, when crushed together by meeting a colourless portion at rest, losing their colour.

If, before any colour appears, we blow upon the film, it produces colourless eddies, like those shown in Figs. 9 and 10.

When the colours are produced on plane or curved films, the streams are singularly beautiful, assuming the most extraordinary shapes, and running from every part of the circumference. The coloured lines are sometimes serrated and sometimes mottled, black portions and portions of rectilineal bands occasionally appearing.

In some films *red* and *green* colours appear the instant they are made, and when this happens, the streams above described are more quickly formed.

When the solution that exhibits these phenomena is diluted with an equal

quantity of water, the film which it gives is much smoother. Two or three serrated bands appear at its apex when it is placed vertically, and the streams of colour flow quickly from the circumference. In one of these films the singular black figure shown in the annexed diagram was produced. The film immediately burst before exhibiting any colours.

When the film from this solution is first made, it is almost always perfectly colourless, and remains so for a short time; but if we blow on it in this state, through a narrow tube, along a diameter or otherwise, the most brilliant systems of revolving rings are produced, as in Figs. 9 and 10, a result obtained from the soap-film only when the colours had previously existed upon it. The first rings that are seen in this experiment are colourless. The highest order of colours then appear in the centre of the system which is most influenced by the blast, and this is succeeded by the next lower order, till the centre becomes black, and the film bursts. When the blowing ceases, the colours often wholly disappear, the colouring matter being restored by a gradual rise of tint to its colourless state, as it was in the newly-made film. If we place the film vertically, after it has been blown upon, an approximation to mottled bands appears, the greater part of the film being colourless. If we blow upon the newly-made film with the mouth, instead of a tube, no colours are produced.

V. *On the Origin and Development of the Colours on the Soap-Bubble.*

It is impossible to witness the simplest of the preceding experiments without being convinced that the common theory of the colours of the soap-bubble is incorrect, and that they are not produced by different thicknesses of the elastic film which composes the bubble. A colouring matter of a very peculiar kind floats upon the bubble, or upon the film, as oil does upon water, or as the oil of laurel varnish of the late Mr DELARUE does, in producing those magnificent colours which he succeeded in transferring to paper. In these cases, however, the colours are caused by the mere expansion of the oil or varnish into thin films, producing what NEWTON calls the colours of thin plates; but in the case of the soap-bubble the colours are formed by minute molecules, either of the soapy solution itself, or, what is more probable, by some of its ingredients or elements expressed or secreted from it only when in the state of a film, forming, under the influence of gravity and their mutual attraction, the different orders of colours we have described.

This curious process, of which there is no other example in the production of colours, may be traced experimentally with the microscope from the formation of the elastic film till its explosion; but by what process the colourless secretion parts with the colorific molecules which compose it, and by what laws and affinities these molecules take their place in the different orders of colours, enter into those

innumerable configurations which we have described, and finally return into their original colourless condition, are problems of great interest and difficulty.

When we examine the surface of a solution of soap, or of the glycerine mixture, in a vessel however shallow, no colour or colouring matter appears upon it. It reflects the images of objects, like the surface of water or glass; but as soon as it is blown into a bubble or formed into a film, its surface becomes for a time uneven, and reflects objects imperfectly, in consequence of its being covered with round-edged waves or tadpole portions of a colourless fluid moving from the margin of the film. When the film is held vertically, these portions rush chiefly to the circumference of the film, and, rising in streams of considerable velocity, give out molecules of different colours, and consequently of different sizes, to form the horizontal bands, carrying the smallest to the apex, to form the black of the first order, and those of a greater size to the bands of the higher orders.

When the colour molecules have thus arranged themselves, they may be scattered, as already mentioned, by blowing, or by pouring upon them some of the soapy solution, or even by brushing them from their place by a feather wet with the solution. Thus scattered over the film, they are singularly mixed together, so as to produce compound tints; but when the film rests in a vertical or horizontal position, they re-arrange themselves under the influence of gravity, taking their place in bands or circles as above described.

The mode in which these changes take place, and by which the horizontal bands are broken up, as shown in Fig. 14, is well seen when a portion of one or more bubbles adhere to the margin of the film, as represented in Fig. 15. When the bands of three or four orders are produced in this film, held vertically, the rush of coloured molecules, in the tadpole form, from the margin *mn* of the bubble B, disturbs, or effaces, as it were, the regular bands; and when the bubble is burst, or bursts spontaneously, it leaves a film behind it, enlarging the original film, and scattering all its colouring matter over the enlarged film. When there is a series of bubbles round the margin of the film, the effect of their bursting, or being burst, in succession upon the successively enlarged film is very beautiful. In one of these experiments the accidentally symmetrical position of four equal bubbles round a perfectly square central film, as shown in Fig. 16, was so remarkable as to deserve being noticed. On all these bubbles there were coloured rings, or rather curved bands, the colouring matter of which descended into the line of junction of the bubbles with the film, and rose to obliterate or disturb its horizontal bands.

In the preceding experiments the soap-film was fixed by capillary attraction to the rim of a wine-glass; but the general phenomena, and especially the formation and breaking up of the coloured bands, may be best seen upon plane films which are surrounded with other plane films, as in the figures of equilibrium discovered by Professor PLATEAU. When an elliptical plane film is formed between two

wire rectangles crossed at right angles,* the four curved films in which it is encased pour upon it their colouring matter in the form of tadpoles, and speedily efface the horizontal bands which are formed upon it when held vertically. The streams of colouring matter which ascend to the apex of the film, along its delicate fluid rim, are seen to much greater advantage than in films adhering to a wine-glass or any other solid body. They are not impeded in their ascent by capillary attraction, and, in general, the currents are in succession of the same colour and order.

There seems to be no direct means of determining the nature of the colouring matter thus active upon the surface of the soap-film, but there is one fact which may prove useful in such an inquiry.

When a convex film was stretched across a conical wine-glass, and had stood ten minutes, it gave brilliant coloured rings of the first and second order. A lens, with its lower side concave, was placed above the film, and happened to be very near it. When the rings had increased to the third and fourth order, the film rose up, as if by attraction, or more likely from a slight increase in the temperature of the air within the glass, and broke upon the concave surface of the lens, leaving a ring of fluid 1·37 inches in diameter, and within it a distinct colouring matter, which continued visible for several hours.

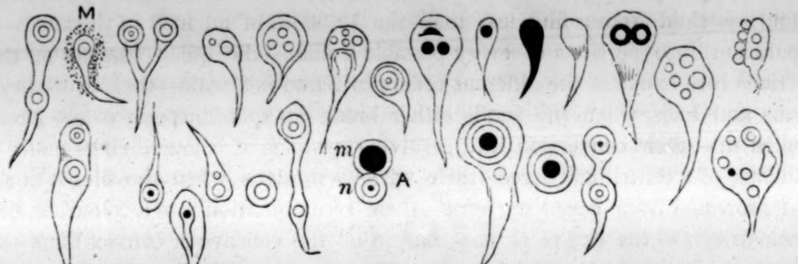
In order to see more distinctly the coloured rings thus embalmed, I covered the concave side of the lens with black wax, so as to reflect as little light as possible, and repeated the experiment. The film again rose and burst, and the colouring matter remained upon the wax *thirty-two hours.* Even when the wax is at some distance from the film, the experiment will always succeed by heating the air within the wine-glass, and increasing the convexity of the film till it touches the wax. The electrical excitation of the glass by friction will doubtless answer the same purpose. Whatever, therefore, be the nature of the colouring matter left on the wax, it cannot consist of a number of thin plates of soapy water.

These various phenomena may be finely seen and shown to others by placing the film in a cone of divergent light. The transmitted beam will exhibit the colourless fluid in the act of secretion from the film, rushing up in tadpole and other forms from its circumference to its apex, while groups of circular portions with dark edges and white centres descend from the upper part of the film. The horizontal colour bands while thus forming, and after being formed, will be seen magnified in the reflected beam, and may be farther magnified if we reflect them from a convex film.

In the formation and decay of the coloured bands, the portions of the coloured matter, which I have called tadpoles, perform a curious part. Their movements, whether we examine them singly, or when revolving in one or more masses, while other parts of the film are at rest, are very extraordinary; and their mutual actions, and apparent vitality, are so unlike anything that has been

* *See* the following paper.

hitherto described, that a more careful study of them may enable us to form some idea of their structure, and of the matter of which they are formed. A very few of their numerous and ever-changing forms are shown in the annexed diagram, and in Figs. 14 and 15. Their outline is generally well defined; and when they



are ascending through the colouring matter of any band they displace it, raising a little its tint at their margin, as at M. The breadth of their head seldom exceeds the thirtieth of an inch, and it increases with the size of the film. They carry in their heads, and also in their tails, colouring matter of various tints and orders; and when borne to the top of the films, as at A, by the lighter matter *m* of the first orders, they leave it in its place, and return with their heavier freight *n* to deposit it in the bands to which it belongs. Single tadpoles are often united together in the most capricious manner, both laterally and by their tails; and when the whole film is covered with them and without bands, they will lay themselves together, so that a band of a particular colour will be formed by their similarly-coloured tails, and another adjacent band by their similarly-coloured heads.

When a plane film is slightly inclined to the horizon, a number of colourless circular portions and colourless tadpoles will at first move over its surface, rising to its apex, chiefly round its circumference. These tadpoles gradually show colours of different orders, which are carried to their appropriate bands. A large portion of this colouring matter is frequently carried up the margin of the film to its apex, where the black band must be formed; but, as it cannot remain there, it moves slowly downward, reinforcing the bands to which it belongs, and leaving the apex for the formation of the black band.

When the black band is nearly as large as in Fig. 5, and there is colouring matter above it, it will cut its way through the black mass, dividing it into three or four portions, separated by narrow channels, through which the colouring matter moves down in small specks to join their proper bands. When this is done, the three or four portions of the black band unite, and the band expands itself over the whole film, pushing before it, in striped currents or in circular portions, the higher orders of colouring matter, which returns to its original

colourless state, as shown in a former experiment, where, in a concave film, it was collected into a double convex lens, in the centre of the black film.

It is impossible to convey in language an adequate idea of the molecular movements and the brilliant chromatic phenomena, exhibited on the soap bubble; and it is equally impossible for art to delineate them. The visible secretion of a colourless fluid from a film less than the 12,000th of an inch in thickness—its separation into portions of every possible colour—the quick passage of these portions into bands of the different orders in NEWTON's scale—their ever-varying forms and hues, when the bands either break up spontaneously or are forcibly broken up—their conversion into revolving systems of coloured rings under the influence of a centrifugal force—their various motions when the film is at rest, and protected from aerial currents—their recombination into a colourless fluid, when driven to the centre or the margin of the concave or convex films—and their reabsorption by the film by means of mechanical diffusion—are facts constituting a system of visible molecular actions, of which we have no example, and nothing even approaching to it in physics.

The phenomena of colour, described in the preceding experiments, are more various and beautiful than any I have ever witnessed, whether caused by refraction, or by thin plates under the influence of common or polarised light. The compound tints produced by ever-varying changes in the combination of the differently-coloured molecules, have a brilliancy and a peculiarity of hue which I have never before observed; and I am persuaded that, if we could examine them prismatically under the microscope, we should obtain remarkably banded spectra.

Owing to the small size of the soapy film, it is not easy to show these colours to more than one individual, but it would not be difficult to exhibit them by the magic lantern to the largest audience.

V. *On the Mode of Producing Plane and Curved Films, and Examining the Phenomena they Exhibit.*

In conducting experiments on the colours of the soap-bubble, it is of great importance to be able to produce films perfectly plane, and films with various degrees of convexity and concavity, to protect the colouring matter upon them from currents of air, and get rid of the extraneous light by which the phenomena may be obscured.

1. *Plane Films.*—Plane films are often produced by dipping the rim of a cylindrical wine-glass in the soap solution; but they are frequently convex, and sometimes concave, results which I cannot explain.

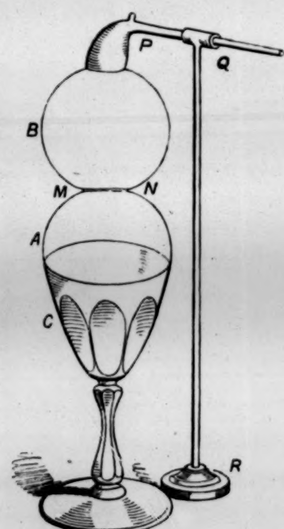
Surfaces almost perfectly plane are invariably produced by using cylinders of glass or metal, open at both ends. They are more perfect when the metallic rim is smooth and accurately circular.

A good plane film may be produced in a very singular way. If we deposit a bubble on the mouth of a cylindrical or conical wine-glass, a little less in diameter than that of the bubble, the bubble, according to its size, will leave on the glass one-third, or one-fourth, or one-fifth of itself as a plane film, and will stand above the film two-thirds, three-fourths, or four-fifths of a sphere.

If we deposit the same bubble upon a cylinder of glass or metal, open at both ends, it will deposit the lower portion of itself as a concave film upon the cylinder; and if we burst the bubble, the concave film will start into a plane one. Another method of producing perfectly plane films has been already described.

2. *Convex Films.*—A convex film is frequently produced upon a cylindrical wine-glass, but always upon a conical one; and we can easily convert a plane or concave film, when formed upon a *closed* cylinder, into a convex one, by heating the air within the cylinder. In some cases I have thus converted a plane film into nearly a complete sphere. The same result may be obtained when the film is at one end of a long *open* cylinder, by plunging the lower end in water.

3. *Concave Films.*—Films of this kind are less easily obtained than those which are plane and convex. They are often produced upon a cylindrical wine-glass, as has been already stated; and they may be always produced by depositing a bubble upon the end of an open cylinder.



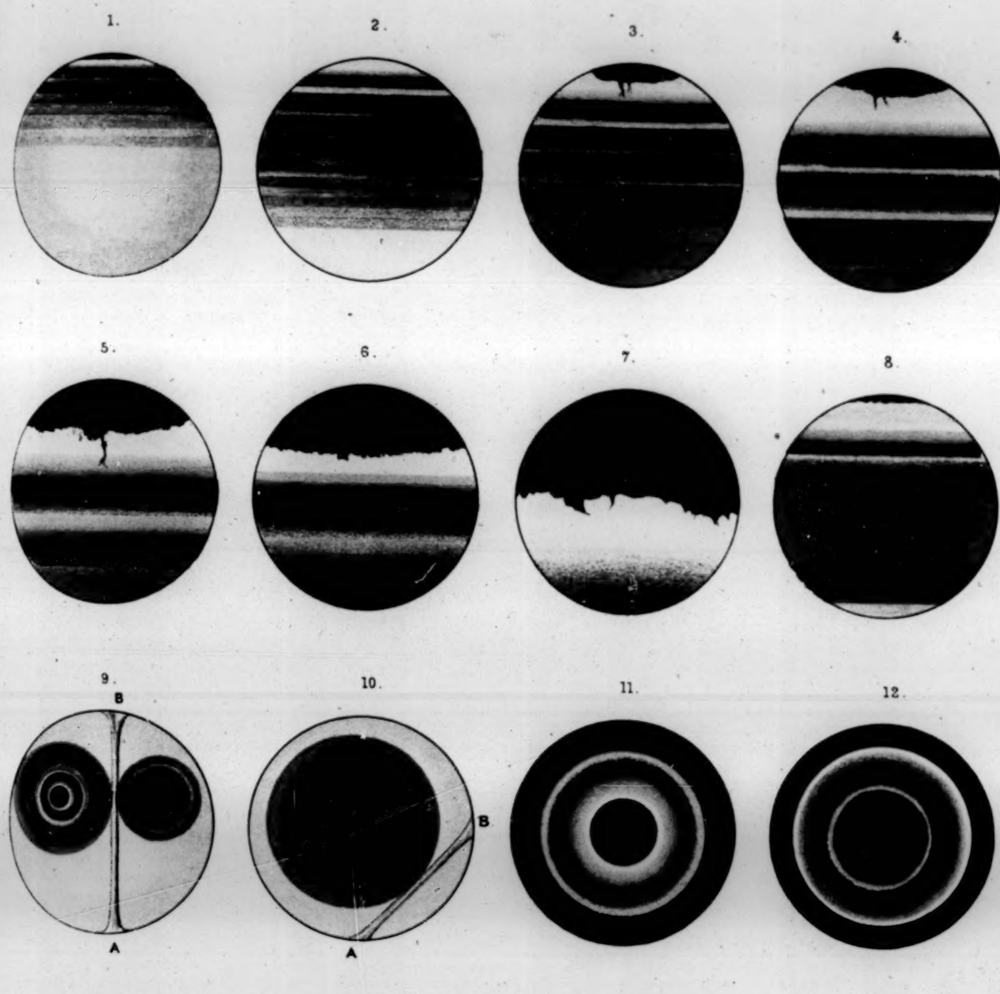
4. *Plane, Convex*, and *Concave Films.*—All these films may be obtained in succession by the juxtaposition and partial union of two soap-bubbles of the same or different sizes.

Let A be a bubble, deposited upon a wine-glass C or an open cylinder, and B another bubble laid upon A, and kept there by the pipe PQ, supported upon a stand QR. The two bubbles will be separated by a film MN. If the bubble A is equal to B, the film MN will be plane. If A is *greater* than B, MN will be *concave;* and if A is *less* than B, MN will be *convex*. If, when A is *equal* to B, we enlarge B by blowing through QP, MN becomes *convex*. If we then diminish B, by sucking out the air at Q, MN becomes *concave*. In all these three forms the film MN is competely protected from air, and almost completely from extraneous light.*

* After I had used this method of producing the three varieties of films, I found that M. Plateau had long ago discovered the relation between the size of two united soap-bubbles and the curvature of the film which separates them.

In protecting the colouring matter upon films from currents of air, I at first employed watch-glasses and plates of parallel glass, but the protection was not complete; and the phenomena were obscured by the vapour deposited upon them, and by the light reflected from their surfaces. When I discovered the method of forming films beneath bubbles, and by the union of bubbles of different sizes, I was able to protect the colouring matter completely from currents of air, and at the same time to get rid of the aqueous vapour and the reflected light.

In examining films produced upon wine-glasses or other bright substances, the brilliancy of the colours is greatly impaired by the light which these substances reflect; but if we cover the inside of the glasses with black varnish, or blacken it with smoke, we remove entirely the reflected light, and give great distinctness and brilliancy to the phenomena.



W. H. McFarlane, Lith^r Edin^r

XXXIV.—*On the Figures of Equilibrium in Liquid Films.*

By Sir DAVID BREWSTER, K.H., F.R.S. (Plates XXXIV., XXXV., XXXVI.)

(Read 4th February 1867.)

In repeating some of the beautiful experiments of Professor PLATEAU, on the Equilibrium of Liquid Films, contained in seven Memoirs, published in the "Transactions of the Royal Belgic Academy,"* and in prosecuting my experiments on the colours of the soap-bubble, I observed several new phenomena which may have escaped the notice of the Belgian philosopher.

In plunging a wire cube in a solution of soap, and lifting it up vertically, Professor PLATEAU found that there was formed within it a polyhedron, as shown in Plate XXXIV. Fig. 1, consisting of *twelve* similar liquid films adhering by capillary attraction to the twelve wires which compose the cube, and a small quadrangular film suspended in the middle of them. In many cases M. PLATEAU found that the vertical quadrangular film was often horizontal, as in Fig. 2; and M. VAN REES discovered, that by blowing very lightly upon one of its sides it was reduced to a simple line, and then reproduced in a horizontal position, from which it could be blown again into a vertical position, as in Fig. 1.

Considering that a system formed of twelve films joined at the centre of the cube, as shown in Fig. 3, would, on account of its perfect symmetry, be a system of equilibrium, M. PLATEAU was surprised to find that it could not be produced and rendered stable, without introducing something solid into the system. To do this, he stretched a fine iron wire from one summit of the wire cube to the opposite summit, and having, after immersion, withdrawn it from the glycerine solution, there was at the central point a small quadrangular film which slowly got less and less, till the system in Fig. 3 was produced. M. PLATEAU subsequently found that this system was permanent, when a small drop of the glycerine solution was accidentally retained at the centre of the system. As this drop could exist only as a sphere, it is not easy to understand how the twelve triangular films could be united with it at their apex, so as to produce a state of stable equilibrium.

In repeating these interesting experiments I found that the polyhedron, with a horizontal quadrangular film, Fig. 2, was always produced in the solution I employed, and by blowing through a narrow tube, either upon the edge or the middle of it, it was changed, as VAN REES found, into a vertical quadrangular film, as in Fig.1; but by a longer blast, it passed into the unstable system of Fig. 3,

* Memoires de l'Academie Belgique, tomes xvi. xxiii. xxx. xxxi. xxxiii. and xxxiv.

and after continuing a very short time in that state, it returned into the system of Fig. 1, and then into the original system of Fig. 2. In one experiment it became permanent, as in fig. 1, and did not pass into Fig. 2. In several experiments the system in Fig. 3 remained permanent, with a drop of fluid in the centre, as observed by PLATEAU, but upon examining it with the microscope, it turned out to be *a hollow cube*, which, as we shall presently see, forms with twelve films a system of stable equilibrium.

If we make these experiments with a wire rhomb, the same results will be obtained with such differences as might be expected from the inclination of the films to each other.

When M. VAN REES had produced the normal polyhedron shown in Fig. 1, he happened to dip the lower part of the wire cube into the solution to the depth of some millimetres, and upon lifting it out again, he obtained the beautiful system shown in Fig. 4, where the quadrangular film is replaced with a hollow cube, all the faces of which are curved. When the wire cube is thus dipped into the fluid, a film, as M. PLATEAU observes, is formed on its lower square. This film imprisons the air between itself and the oblique faces of the polyhedron immediately above it, and rising, forms the hollow cube shown in the figure.

As the new film which produces this effect is coincident with the lower face of the wire cube, the hollow cube which it generates must be invariable in size, containing the same quantity of air which is imprisoned in the lower portion of the polyhedron; that is, the contents of the hollow cube must always be equal to one-fourth of the contents of the wire cube.

In repeating this experiment, which does not always succeed, I discovered a general method of introducing into the normal polyhedron hollow cubes of any size, from the smallest to a size which nearly fills the whole interior of the wire cube. This is done by blowing a bubble of the requisite size, and placing it in the central quadrangle of the polyhedron. The bubble instantly starts into a hollow cube, containing the same quantity of air as the bubble, obliterating the quadrangular film, and forming a system of perfect equilibrium. If the cube is smaller than we wish it to be, we can enlarge it by introducing another bubble. As hollow cubes of every size form a system in perfect equilibrium, we see the reason why the microscopic cube, already mentioned, kept the system shown in Fig. 3 in stable equilibrium.

By the same method we may introduce a second hollow cube beside the first, displacing it from the centre of the polyhedron, and, along with it, taking up a symmetrical position of equilibrium, as shown in Fig. 5. The two cubes are not necessarily equal, but when they are so, the side common to both, and passing through the centre of the polyhedron, is perfectly plane, while all the other sides are curved. The second cube may be inserted on the right or left side of the first, as well as above or below it; but it sometimes happens that the bubble

intended to produce it passes into the first cube and enlarges it, instead of taking its place beside it.

This method of inserting *one* or *two* hollow solids of any size in the figures of equilibrium of liquid films, may be extended to the different figures discovered and described by Professor PLATEAU. I have applied it successfully, as shown in Figs. 7, 10, 11; Figs. 6, 8, 9, 12, being the figures given by PLATEAU; and also, as shown in Figs. 14, 15, to a remarkable system of wires, shown in Fig. 13, &c. &c. which had not previously been the subject of experiment.

In Fig. 6, where the wires form a tetrahedron, the bubble is introduced at the centre where the four films meet, and the double three-sided hollow figure which is thus produced is shown in Fig. 8.

In Figs. 8 and 9, given by PLATEAU, where the wires form a quadrangular pyramid, the bubble is introduced where the vertical line or film joins the other four, and the single and double quadrangular figures thus produced are shown in Figs. 9, 10, and 11.

In Fig. 12, given by PLATEAU, where the wires form two rectangles cutting one another at right angles, a plane film, of nearly an elliptical shape, is formed in the centre. It is inclined 45° to the planes of the rectangles, and is attached by each of its sides to twocurvilineal films, which adhere to the four vertical wires. The figure is thus composed of *five* films, *one* plane and *four* of a singularly curvilineal form, as in Fig. 13. The hollow quadrangular figures produced by introducing one or more bubbles into the centre of this elliptical film, or between the curvilineal films, in Fig. 13, is shown in Figs. 14 and 15.

In all these systems of films, when two hollow figures are united, whether they be spheres, lenses, cubes, or any other irregular figures, the film which unites them is *plane* if the contents of the two hollow figures be equal, and *concave* and *convex* if their contents are unequal—the *convex* side being always within the largest figure.

Professor PLATEAU does not appear to have studied the singular effects produced by constructing the system of wires, in Fig. 12, so as to vary the angle formed by the two rectangular planes. With such a movable system we see at once why the elliptical plane is formed between two of the pairs of right angles, rather than between the other pair, for it must necessarily appear between the pair whose angle is greater than 90°; and as it is impossible to join the rectangles with mathematical accuracy at right angles to each other, the oval plane should appear only between the wires whose inclination is greater than 90°. This is true only when the rectangles are lifted perpendicularly out of the soap solution; for if they are lifted out obliquely, the oval film will be formed in the angle which is uppermost, whether that angle is greater or less than 90°, provided that the difference of the angles is not great. In the normal position of the rectangles perpendicular to each other, the major axis of the

elliptical film is about four times greater than its minor axis; but if we increase the angle of the planes, the minor axis will gradually increase till it becomes equal to the major axis, the oval plane becoming rectangular when the planes are inclined 180°, a position which cannot be experimentally obtained. While this change is going on, the four curvilineal films, to which the sides of the elliptical plane are attached, are gradually diminishing, and disappear at 180°. During this expansion of the elliptical film, it is not stretched and made thinner, because it appropriates to itself the fluid of the *four* curvilinear films, which at 180° it extinguishes. If we now diminish the angle of the rectangles, the enlarged oval plane will gradually become more elliptical, giving back its fluid to the four curvilineal planes, till at 90° the plane film resumes its normal size.

If, when in this position, we *diminish* the angle of the rectangular planes, the *minor* axis of the elliptical plane will gradually *diminish*. At 45° it will become a straight line and disappear. The elliptical plane will start into the angle of 135°, pushing towards their wires the four curvilineal films, placing itself between them, and in an enlarged state appropriating a portion of their fluid.

Remarkable as these phenomena are, there is one still more remarkable, which requires the testimony of the eye to make it credible. If in the normal or rectangular position of the rectangles we blow upon the oval film, or between the curvilineal ones, a bubble of the proper size, it will replace the system of films with a hollow curvilineal cube, the sides of which will project beyond the faces of the vertical cube, which, having plane faces, would not project beyond the wires. Within the four triangular spaces at the upper side of this cube, will be four summits where the black spot of the first order in NEWTON's scale will be produced, and at which the bubble will burst. If we now hold the wires vertically, the cubical bubble will burst, and *the system of liquid films which it expelled will reappear, as if it had left its ghost behind it, to recover the elements which the bubble had appropriated!* When the wires were held horizontally this resurrection of the normal system of films did not take place, owing, I presume, to the bubble not bursting symmetrically. The same results will be obtained when the inclination of the rectangles is above 90°.

If we introduce the bubble into the system of wires when empty, the system of liquid films is equally reproduced; but the experiment succeeds better when the bubble is laid upon the oval film, as it thus appropriates the fluid of the different films, and when it bursts there is a greater quantity of fluid for their re-formation. For the same reason, the reproduction of the films is produced if the bubble is burst when it is strongest, by thrusting into it a piece of blotting-paper.

The same restoration of the figures of equilibrium, produced in the tetrahedral and quadrangular systems shown in Figs. 6, 7, 8, 9, and 10, may be effected by introducing bubbles of the proper size, whether they are empty or occupied by their respective films.

These phenomena were so unexpected, that I thought it probable that interesting results would be obtained by the following alterations of the system of wires in Fig. 12.

1*st.* By uniting the upper and lower ends of all the wires, as shown in Fig. 16.

2*d.* By uniting only the upper ends, as shown in Figs. 19, 20.

3*d.* By uniting only the lower ends, as in Figs. 21, 22.

4*th.* By uniting the middle of the wires, as in Figs. 23, 24.

5*th.* By uniting the middle, and also the upper and lower ends of the wires, as in Figs. 25, 26, 27, and 28.

6*th.* By uniting the wires at various points successively from their lower ends, till they reached the position in the second system, as in Figs. 29, 30, 31, and 32.

1*st.* In the *first* system, Figs. 16 and 17, we have *four* vertical and equal rectangles, and *eight* horizontal and equal right angled triangles, each of which is concerned in producing the complex figure of equilibrium, shown in Figs, 16, 17, and 18, and consisting of twenty-one films. As the wires are bound together, we cannot vary the angle of the rectangles from 90° to 180°, in order to see that the same remarkable changes will take place, as by varying the angle in fig. 12; nor can we show, that by diminishing the angle from 90° to 45°, the central film will become a line, and then start into the angle of 135°, but we can prove it by uniting the wires when the rectangles are variously inclined to each other.

In the normal state of this system, and in its various conditions from 90° to 180°, two of the right-angled triangles, and two at the bottom in front of the plane cen tral film, contain a plane liquid film,as in Fig. 18,* and the other two ahollow triangular pyramid of films. When the central film is reduced to a line, and at 45° should start into the angle of 135°, the law of equilibrium demands that the four plane films should sink into triangular pyramids, and the four triangular pyramids rise into plane films!

If we now introduce bubbles of different sizes into the different angles of this system of wires, the figures of equilibrium are very various and beautiful, changing with the inclination of the rectangular wire planes, and the size of the bubble.

2*d.* When the rectangles are united only at their *upper* ends. In this case, the system shown in Figs. 19, 20 consists of thirteen films, the central one of which is a semi-ellipse nearly, with an angular summit upon its minor axis. The four films adhering to this film, and the vertical wires, are curved, and the remaining eight are the same as those shown in Fig. 18.

3*d.* When the rectangles are united only at their *lower* end. In this case the system shown in Figs. 21, 22 is the same as that in Figs. 19, 20, *inverted.*

4*th.* When the *middle* of the rectangles only are united. In this case the system shown in Figs. 23 and 24 consists of *sixteen* films—those in the upper half being similar to those in the lower half.

* Fig. 18 is an enlarged vertical view of the upper face of figs. 16, 17.

5th. By uniting the *middle* and the *upper* ends of the wires. In this case the system shown in Figs. 25, 26 consists of *twenty-four* films.

6th. When the *middle* and the *lower* ends of the wires are united. In this case the system shown in Figs. 27 and 28 consists of *twenty-four* films, and is the preceding one inverted.

7th. When the wires are united at several points successively from their lower ends, till they approach the position of the second system, in Figs. 19, 20, the fluid never reaches the four triangles at the top. In this case the films are *eight* in number in Figs. 29 and 30, and *nine* in Figs. 31 and 32. The *curved* films in Fig. 29 are changed into *plane* ones in Fig. 30 by raising the movable wire; andby raising it higher, a portion of a plane elliptical film is added beneath the quadrangular pyramid, and this film increases in altitude till the movable wire reaches the top, when the whole figure is similar to that in Figs. 19 and 20.

Professor PLATEAU has given the figure of equilibrium in an equilateral triangular prism when its height is equal to one of its sides. It is very simple, consisting only of eight films, as shown in Fig. 33. By dividing its height by means of a movable equilateral triangle, I have obtained very curious figures. When the movable wire bisected the height of the prism, I obtained two different figures—the one as frequently as the other, and yet they had not the slightest relation to each other. The most careful adjustment of the movable triangle did not help me to determine which of the two belonged to the bisection. I found, however, upon dividing the height of the prism unequally, that the one figure belonged to the larger, and the other to the smaller half—the prism, in the position of indifference, being guided in its choice of figure by some trifling cause which I failed to ascertain. These figures are shown in Figs. 34, 35, 36, 37, 38, 39, 40, 41, and 42.

Having obtained such curious systems of films by dividing the equilateral prism into portions of different lengths, I was anxious to see the effects produced in the triangular and quadrangular pyramids by wires passing from their apex to the middle of the sides of their base.

The systems of films produced in the triangular pyramid are shown in Figs. 43, 44, 45, 46, 47, where the hexagonal figures are very beautiful. The hexagonal faces are plane, convex, or concave, according as the contents of the hollow figures which they separate are equal or unequal—the face being generally concave. A view of the hexagons, as seen by looking down through the apex of the system, is shown in Fig. 46, while Fig. 47 represents them as seen by looking upwards through the base of the pyramid. The curves of contrary flexure, and the convex ones joining the films adhering to the six vertical curves, and the curved films which they bound, are not easily represented by simple lines.

The systems of films produced in the quadragular pyramid are similar to those in the triangular pyramid, the figures being octogonal in place of hexagonal, as shown in Fig. 48.

The figures of equilibrium of liquid films are finely seen in the union of spherical bubbles and hollow lenses. When two spheres are brought into contact they are united by a film common to both.* When the one is laid above the other, the lower part of the upper one starts into union with the upper part of the lower one, and forms a single circular film. If the upper sphere is retained by the pipe that blows it, it may be rotate upon the other, as upon a joint, without any change in the uniting film. If we now blow through this pipe we can enlarge the upper sphere, and if we suck through it, we can diminish it; so that we can at pleasure make the two spheres equal or unequal. When the upper sphere is equal to the lower, the uniting film is a circular plane. When it is greater than the lower, the uniting film is *convex*, and when it is less, the uniting film is *concave*, the convex side being always turned to the larger sphere.†

When two lenses are united by a film, the film is *plane*, *convex*, and *concave*, according as the one lens is *equal* to, or *greater* and *less* than the other. A double and equally convex lens may be formed at pleasure, as we shall presently see; but I have not been able to bring the lenses into union. They are, however, frequently found in that position in experiments with the wire cube.

The production of plane, convex, and concave films by the union of two bubbles, and the protection of these films from aerial currents by the superincumbent bubble, I found of great use in studying the colours which they produced.

The formation of films, by immersion, upon open and closed vessels of different shapes, their deposition on the same vessels from bubbles, and their motion within certain vessels, which I believe has never been observed, present many curious phenomena.

When a film is formed by immersion on the rim of a closed cylindrical vessel it is generally plane, but sometimes concave, and rarely convex; but in all cases it may be made of any degree of convexity by the application of heat to the vessel. When the vessel is open at both ends the film is always plane.

When the film is formed on a closed cylindrical vessel by the deposition of a bubble upon its rim, it is always plane. The bubble leaves the lower part of itself upon the rim as a *plane* film, but quite separate from the rest of the bubble which stands over it.

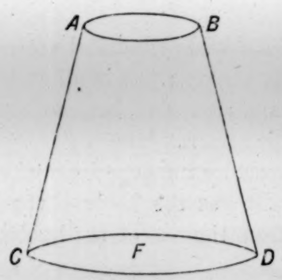
If we use a thick metallic ring with a broad rim, which itself gives a plane film by immersion, a bubble laid upon it deposits a *concave* film, which at first appears to be the lower end of the bubble, but it is quite independent of it in position, as the bubble rests upon the upper and outer side of the ring, while the concave film adheres to the inner and lower end of the ring. They are, however, so related to one another, that when the bubble bursts the concave films starts

* See the preceding paper, p. 503.

† The relation between the size of the spheres and the curvature of the uniting film was first observed by M. Plateau.

into a plane. When the diameter of the bubble is equal to that of the ring, the ring will form its equator, and the bubble will be a perfect sphere passing into a plane film when burst. If, when the bubble is a complete sphere, with its metallic equator, we place another upon it, and alter its size by blowing and sucking through the pipe that holds it, we shall observe the production of *plane, convex*, and *concave* films, which has been already mentioned. When the bubble bursts, a double hollow convex lens is often formed with particular solutions, though a plane film is most frequently the result.

When a bubble is placed upon the mouth of an open cylinder like the chimney of a lamp, the glass being quite dry, it will deposit a film, which will immediately move down the tube about an inch, and the bubble will burst, leaving another film in its place. Both these films sometimes remain, the *uppermost* being *convex* and the other *concave*. If we close the lower end of the tube by plunging it in a tumbler of water, a bubble laid upon its mouth will deposit a plane film there. If we now lift the tube slowly, the plane film will descend, becoming concave before it bursts. By continuing to lift the tube the truncated bubble will grow less and less, till it becomes a plane film about the eighth of an inch below the rim of the tube. If we now depress the tube the film will gradually rise to the primitive bubble. When the bubble is small it descends about one or two inches as a convex film. If the diameter of the primitive bubble is a little less than the diameter of the tube, it generally deposits itself within the tube in the form of a semicylindrical film.



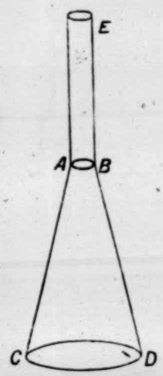
In all conical tubes closed at one end, the film, taken up by immersion, or deposited by a bubble of considerable size, is always convex. The film is also convex in closed cylindrical tubes if the mouth is slightly widened as in test tubes. If the bubble deposited upon cylindrical tubes of this kind is very small, but not less in diameter than their own, the film is plane, and sometimes slightly concave, descending a little in the tube. If the bubble is smaller still, it forms itself into the semicylindrical film already mentioned.

If we employ a truncated cone the phenomena and motion of the films are very remarkable. When the cone ABDC is closed at CD, the other end, AB, gives by immersion a *plane* film. When a bubble is laid upon AB it forms a concave film. Upon admitting the air gradually at CD, the concave film descends, *expanding* itself into a larger plane film which breaks at CD. The bubble then follows it, leaving a plane film at AB. When the cone ABCD is closed at AB, it gives, by immersion, a *convex* film, which by the gradual admission of the air at AB descends as a plane film contracting itself to AB.

If, when both ends of the cone are open, we take up, by immersion, a film upon the larger side BC, it will rise to the top AB, gradually *contracting* itself, and remaining at AB. If there happens to be a drop of fluid retained at F, its weight will prevent the film from rising, but if we run off the drop F at one side, the film will immediately rise to AB. The height of the glass cone ABDC is $1\frac{1}{2}$ inch, its breadth AB 1·1 inch, CD 1·7, and its angle 13°.

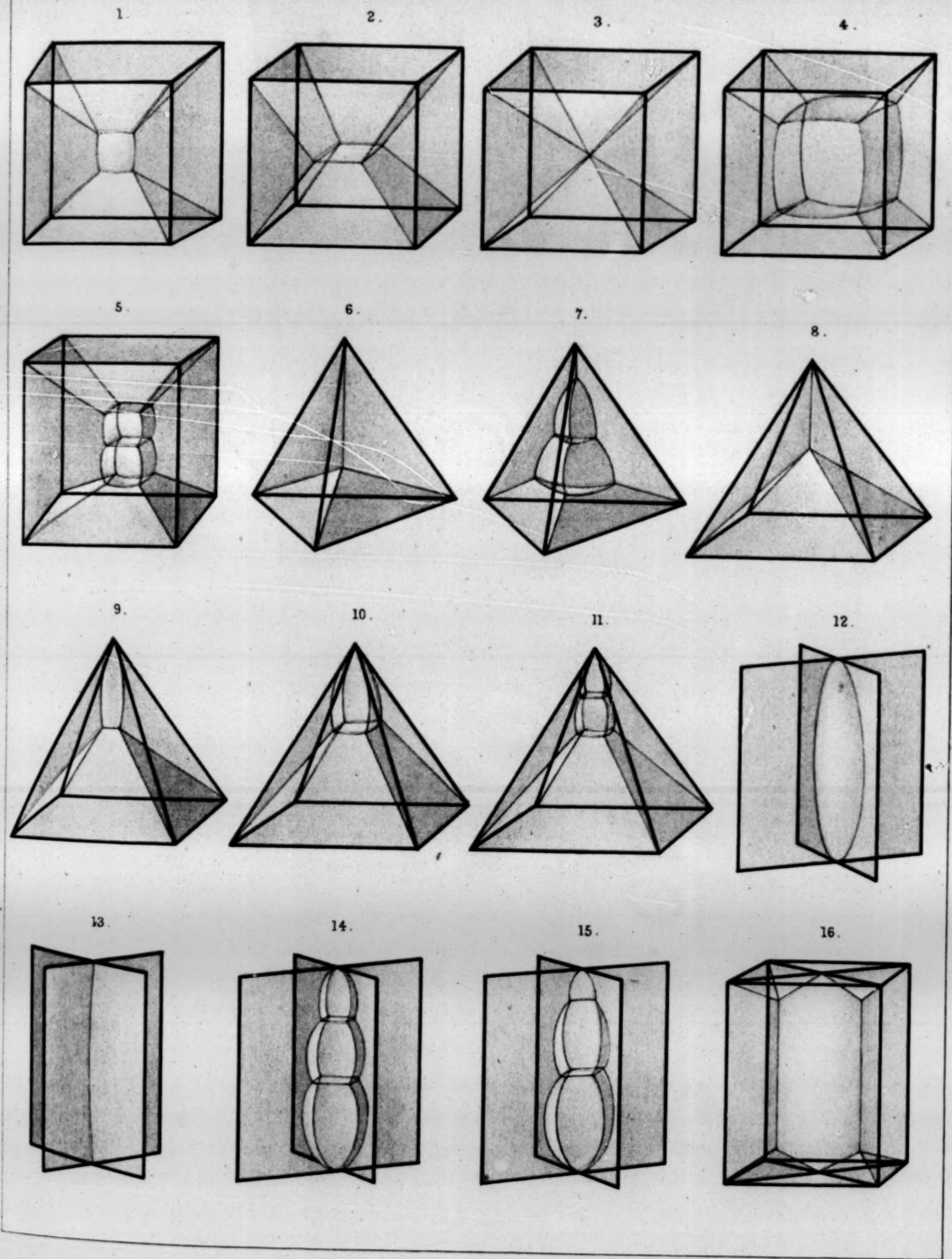
When the bubble, placed at AB, is not much larger in diameter than AB, it will descend with the film, leaving another film at AB. Upon breaking AB the film CD will rise to AB.

If we employ a glass funnel ABCD of the same size and form as the preceding figure, and close it at E, the film which it takes up at CD will be *convex*, but if we open it at E it will take up a plane film which will ascend, breaking at AB, and leaving a film there. If we now take up another film by dipping CD the eighth or tenth of an inch in the solution, the compression of the air will drive the film AB out of the tube at E. If we replace AB by making a film at CD ascend to AB, and dip CD into the solution so slightly as not to compress the air and drive out AB, we shall obtain a *convex* film at CD, which will remain there. By breaking AB with a strip of blotting paper, the *convex* film CD, with its coloured rings, will become plane and rise to AB, the rings formed on the convex film being changed to bands when the film has become plane.



I was anxious to ascertain what was the largest angle of a cone at which it would raise the film at its mouth to the aperture at its apex. Upon using cones made of thin card-board, I found that the film rose rapidly when the angle was upwards of 90°, but when it reached 120° or 130°, a film could not be lifted from the soap solution. It was probable, however, that the cone did lift the film and carry it to its apex, as there was always the appearance of fluid at that place. By placing the solution in a vessel shallower than the height of the cone, and observing that there was no film at the apex previous to lifting it out of the fluid, I always found that a film was at the apex, and consequently that a film had been formed at the mouth of the cone, but rushed so rapidly upwards that it could not be seen there.

Plate XXXIV.



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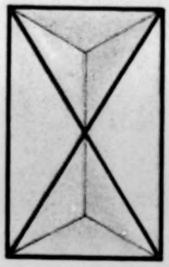

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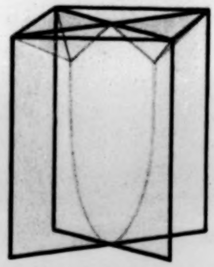

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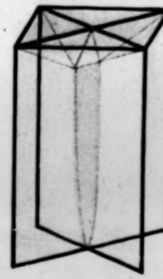

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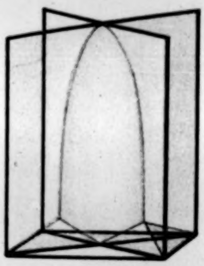

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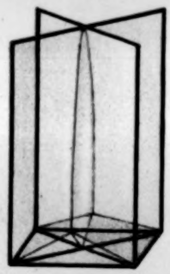

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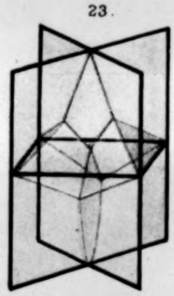

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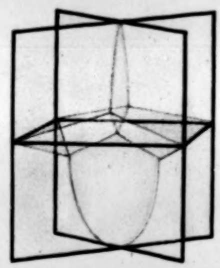

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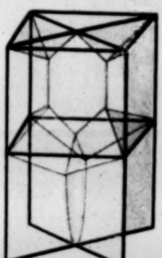

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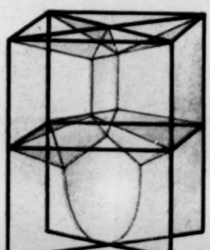

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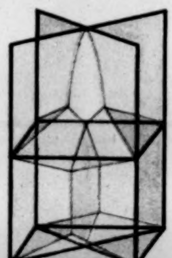

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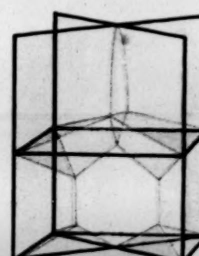

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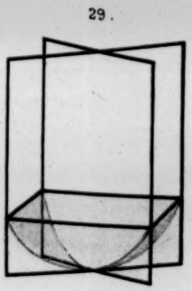

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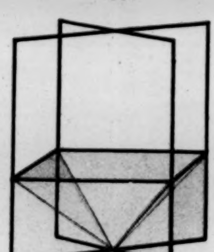

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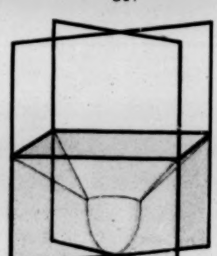

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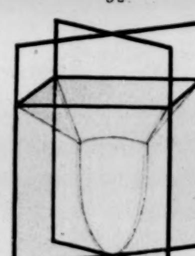

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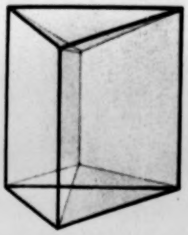

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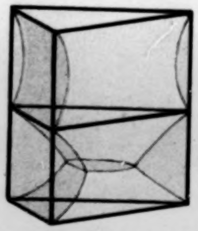

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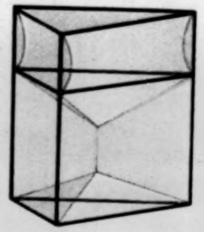

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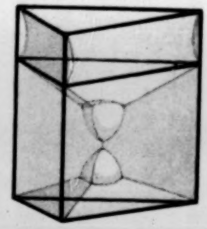

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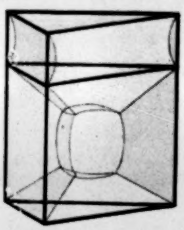

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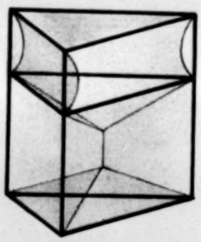

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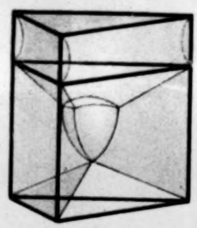

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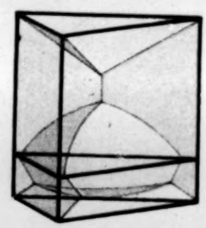

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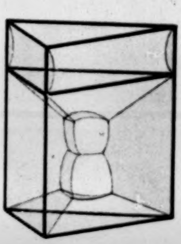

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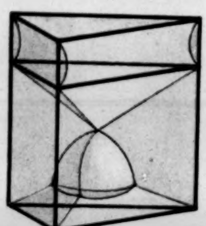

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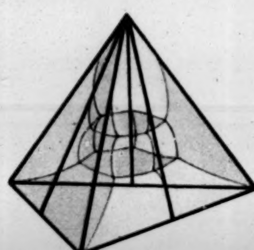

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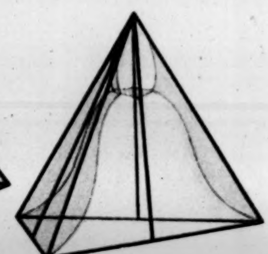

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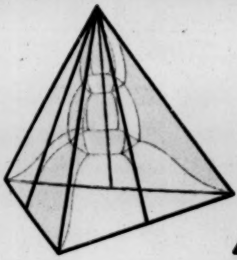

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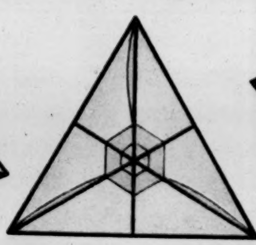

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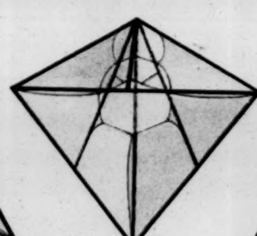

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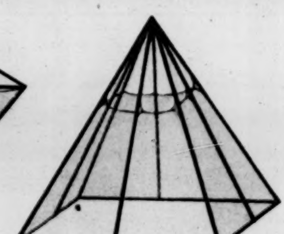

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XXXV.—*On the Third Co-ordinate Branch of the Higher Calculus.* By Edward Sang, Esq.

(Read 15th January 1866.)

That part of universal arithmetic which relates to the changes of variable quantities, and which is known under the titles—*Fluxions, The Higher Calculus, The Infinitesimal Calculus, The Theory of Functions*, has been divided into two branches, called respectively the *Differential* and the *Integral Calculus ;* the one of these being regarded as the converse of the other; and every problem connected with variation has been supposed to require either or both of the processes known as *differentiation* and *integration.*

This two-sided view of the higher calculus arose naturally in the course of its development. When we study the changes of a variable quantity, our attention is called to its differences and to the velocity of the change. The first branch of the subject is thus that which teaches us to pass from the variable quantity to its differential or differential coefficient; and, as our attention has been engrossed by the mutual relations of the variable and the differential, the converse problem "to pass back again from the differential to the integral" seems to be its complete complement.

Just so, in the progress of algebra proper, attention was drawn to the squares and cubes of numbers and to their higher powers. Thereafter it was directed to the inverse problem "from the power to compute the number or root," and thus algebra was divided into two parts treating, respectively, of Involution and Evolution; these two processes seeming to be the complete converses of each other.

However, later arithmeticians, regarding the subject more comprehensively, have considered that there are three related things, the root, the index, and the power, and that, therefore, there are three branches of the subject or three problems—the first, when the root and index are given and the power sought; the second, when the index and the power are given and the root sought; and the third, when, the root and the power being both given, the index is sought. The last of these problems was resolved by Nepair; it is the exponential or Logarithmic problem.

In the present paper it is my object to indicate that the Theory of Variables has, like Algebra proper, a third co-ordinate branch bearing to the differential and integral calculi relations somewhat analogous to those which the Theory of Logarithms bears to Involution and Evolution.

In the process of differentiation, the variable quantity is regarded as a function or dependent of some primary variable, and the ratio of its change to the change of this primary is sought for; this ratio being the differential co-efficient, or, as LAGRANGE calls it, the *derived function*. Here we have three connected variable quantities, the *primary*, the *function*, and the *derivative;* and thus we have three distinct problems characteristic of the three great branches of the calculus.

In the first of these branches the leading problem is this,—having given the relation between the primary variable and its function, to discover the derivative. In the second branch, the relation between the primary variable and the derived function being given, the primitive, or, as it is called, the integral, is sought. While the province of the third branch is to discover the primary, when the relation between the primitive and derivative functions is given. This third branch may thus be called the Calculus of Primaries.

This classification of the different branches of the calculus of variables is complicated by the existence of various orders of derivatives. Besides the three variables quantities, viz., the Primary, the Function, and the Derivative, there is the order of derivation to be considered, and, therefore, it may be argued, there must be four complementary problems, the fourth problem being that in which the order of derivation is the *quæsitum*, the Primary, the Function, and the Derivative, being the *data*. The mutual relations of these problems have been obscured partly by circumstances incidental to the subject, and partly by the peculiarities of the notation employed. The notation used by NEWTON, and revived in a slightly modified form by LAGRANGE, is essentially defective; while that contrived by LEIBNITZ, and now commonly in use, is cumbrously redundant. In order to denote the derivative of a function, LAGRANGE places an accent over it; to indicate the second derived function he places two accents, and so on. The radical defect of this notation is apparent, when we consider that a variable quantity U may be regarded as a function of one or of another primary, and that the symbol $'U$ contains no indication of the primary; thus ^{iv}U is simply the fourth derivative of the variable U. On the other hand, the notation of LEIBNITZ is explicit on this score, the symbols $\frac{d^4U}{dx^4}$ and $\frac{d^4U}{dy^4}$ indicating the fourth derivatives of the variable U, regarded in the one case as a function of x, in the other case as a function of y; these symbols also paint, as it were, the process of derivation; they, however, contain a redundancy of parts: the symbol of derivation is twice written, and so also is the index of the order. Now, essentially, there are four indications to be made; we need the sign of derivation, its order, the function to be operated on, and the primary in regard to which the derivation is to be made. The sign of derivation may be given by a conventional arrangement of the characters, just as in products and powers; there remain, therefore, three things

to be symbolised,—the variable operated on, the primary or argument of which it is regarded as a function, and the order of derivation. The relative positions of these symbols is a matter of mere taste or convenience. I have adopted the arrangement of writing the numeral of the order and the primary variable as ante-subponents to the function—thus I use the formula ${}_{2t}U=Z$ to denote that Z is the second derivative of U, regarded as a function of t. It is impossible to indicate the proposed relation with fewer letters, and any more would be redundant. The same relationship is expressed, according to LEIBNITZ's notation, by

$$\frac{d^2U}{dt^2}=Z.$$

When we desire to indicate the opposite relationship, that is, to state that U is the second primitive of Z, the usual notation is $\iint Zdt^2=U$, whereas by using the ordinary sign of reversion, viz. —, we may indicate the same thing by ${}_{-2t}Z=U$, that is to say, U is the second primitive of Z, regarded as a function of t. In this way the expressions, etc.,

$$_{-3t}x,\ _{-2t}x,\ _{-1t}x,\ x,\ _{1t}x,\ _{2t}x,\ _{3t}x,\ \text{etc.},$$

denote a series of functions of t, each one of which is the derivative of the preceding, or the primitive of the succeeding term.

In the general equation ${}_{nt}U=z$, three variable quantities and the constant number n are combined; and, as I have already said, the most comprehensive form of the problem is "from any three of these to find the fourth." But, since n is necessarily constant, that case of the general problem in which n is the *quæsitum*, must differ essentially in its nature from the other three; nay more, unless the given relations among the three variables t, z, v, be such that z is one of the derivatives or primitives of v, regarded as a function of t, the problem can have no solution, it is indeed altogether unmeaning.

Thus, if ${}_{nt}\sin t=t^3$ were proposed, that is, if it were demanded, "how many times must $\sin t$ be derivated until the result t^3 be arrived at?" our only reply would be, that the question has no meaning, for t^3 is not any derivative of $\sin t$, the only derivatives of this function being $\cos t$, $-\sin t$, $-\cos t$, and $+\sin t$.

To those who uphold the continuity of algebraic expressions, and claim even for the square root of a negative quantity a real existence, this argument must appear quite inconclusive, because, between the members of the above written series, there may be intermediate terms; thus, there may be such a term as ${}_{\frac{1}{2}t}x$ or as ${}_{2\frac{1}{2}t}x$; there may be some yet undiscovered operation, two performances of which may produce the same effect as one derivation; and although such expressions may appear unmeaning to us, they may not be more so than the analogous symbols $a^{\frac{1}{2}}$, $a^{2\frac{1}{2}}$ appeared to the earlier algebraists. It is impossible to assert that some such operation may not yet be discovered, although, at present, we can form no conception of its nature. There may even be processes such as

to give significance to the symbol ${}_{\gamma_0 t}x$; there may be any number of terms interpolated into the above series, and we cannot say that t^3 is not one of the endless numbers of terms between $\sin t$, $\cos t$, $-\sin t$, $-\cos t$.

In a paper which I gave in the *Annals of Philosophy*, for August 1829, it is shown that if u and v be two functions of some primary, which for the sake of conciseness we may suppress, the n^{th} derivative of their product uv, takes the form

$${}_n(uv) = {}_n u \,.\, v + \frac{n}{1}\, {}_{n-1}u \,.\, {}_1 v + \frac{n}{1}\,\frac{n-1}{2}\, {}_{n-2}u \,.\, {}_2 v + \text{ etc.}$$

which is quite analogous to the n^{th} power of the binome $u+v$; and it is also shown, that the n^{th} primitive (or integral) of the same product is

$${}_{-n}(uv) = {}_{-n}u \,.\, v - \frac{n}{1}\, {}_{-(n+1)}u\, {}_1 v + \frac{n}{1}\,\frac{n+1}{2}\, {}_{-(n+1)}u \,.\, {}_2 v - \text{ etc.}$$

which is also the counterpart of $(u+v)^{-n}$.

It is thus seen that these expressions are true whether a positive or a negative value be assigned to n, and the question may well be propounded, "Do these formulæ hold good when n has a fractional value?"

Without entering farther into this subject, it is enough for me to repeat the remark, that in the present state of our knowledge, we can form no idea of such fractional derivation; and that, therefore, the fourth branch of the general problem, viz., that in which the order of derivation is sought for, has for us scarcely any significance. There remain, then, only the three branches, the calculus of differentials or derivatives, that of integrals or primitives, and lastly the calculus of primaries.

The great majority of problems in theoretical mechanics belong to this third branch. The velocity of a moving point is generally dependent on its position; now, the velocity is the first derivative of the position regarded as a function of the time, and thus the problem really is, "from the known relation between the function and its derivative, to determine the primary, viz., the time." This problem is commonly resolved by a very simple process,—the first derivative of x, regarded as a function of t, is the reciprocal of the first derivative of t, regarded as a function of x; and so, by causing t to appear as the function, and x as the primary, we convert the problem into another belonging to the integral calculus. This convertibility of the problem has somewhat concealed its true nature.

The first example which we have on record of the use of the laws of change in scientific research, belongs to this third branch of the subject, and clearly exhibits its true nature. NEPAIR regarded his two flowing quantities, his *Artificialis* and *Naturalis*, as connected by this law that, while the velocity of change in the former is uniform, that in the other is variable, and proportional to the *Naturalis* itself. In modern language he made the *Artificialis* his primary variable, and prescribed the condition that the first derivative of the function should be pro-

portional to that function (Constructio, sect. 26). In his first logarithms, also, he supposes, that at the outset the two velocities are alike. From these premises, he constructed his three auxiliary tables which are really what we now call *antilogarithmic*. After the computation of these tables of *radicales*—after the compilation of his *Constructio*; and, probably, even after the completion of the *Canon Mirificus*, he began to view the matter from the other side, and to think of his system as applicable to calculation in general; that is, he viewed the *Naturalis* as the argument, the *Artificialis*, to which he now gives the expressive name *Logarithm*, as the function. Inverting the relations of primary and function, he states in his appendix, article Habitudines, sect. 2, that the velocity of the increase of the logarithm is inversely proportional to the number (*sinus*). Thus, the genesis and computation of logarithms given by NEPAIR, is a perfect example of the transition from the equation $\frac{\delta x}{\delta t} = \phi x$, which belongs to the calculus of primaries, to the new equation $\frac{\delta t}{\delta x} = \frac{1}{\phi x}$, which belongs to the integral calculus.

The great problem in mechanics, "having given the law of attraction to compute the motion of a body," belongs to the second chapter of our calculus, for the attraction is the second derivative of the position regarded as a function of the time, and thus all mechanical problems of this class are typified by the general formula $\frac{\delta^2 x}{\delta t^2} = \phi x$, or ${}_{2t}x = \phi x$, in which the law connecting the function with its second derivative being given, that which connects it with the primary is sought. In this case also, the problem can be brought back to the calculus of integrals, the artifice being to multiply each member of the equation by the first derivative, and then to obtain by integration the square of the velocity in terms of the position.

The calculus of primaries has thus been unfortunate in that its first two chapters have been absorbed by the Integral Calculus. But this absorption ceases when we have to do with derivatives of a higher order, for then the resources of integration fail us. As yet, no artifice has been discovered whereby the equation $\frac{\delta^3 x}{\delta t^3} = \phi x$, or ${}_{3t}x = \phi x$, can be rendered integrable, and problems involving such equations must be resolved by methods special to themselves; or, as not unfrequently happens, must be let alone.

There are many problems connected with the geometry of motion and with mechanics, which resist all the powers of the calculus; thus, although we know the law connecting the curvature of an elastic plate with its angular tension, that is, although we be able to write down an equation of the shape in differentials, we are unable thence to arrive at that shape; in other words, we are unable to integrate. And thus our progress in mechanical science, even in matters of

daily occurrence, is arrested, not by our ignorance of the mechanical principles involved, but by the imperfection of our attainments in arithmetic.

The integral calculus may almost be described as a collection of artifices by which a great variety of proposed differentials may be transformed into other differentials of which the integrals are known, not by integration proper, but by previous differentation. These artifices must necessarily bring us to known functions, or to combinations of known functions. Now the number of simple functions employed in the calculus is very limited; the usual enumeration gives us the *potence* or algebraic function, the *exponential*, the *logarithmic* function, the *sine*, and the *cosine*, in all five; and even of these five two must be removed, because the logarithmic and exponential functions are converse to each other, while the cosine is only a variety of the sine; and thus we have but three kinds of simple relationship from which we seek to compound all others.

When we contrast this paucity of material with the multifariousness of the work to be accomplished, we need not be surprised that so much remains to be done; it is rather matter of astonishment that so much has been performed. The mutual relations of physical phenomena are too various, too complex, to be represented by any combinations of such a small number of functions; and our hopes of passing beyond the present limits of applicate analysis must be founded on an extension of the groundwork, and on an enlargement of our plan of operation.

In the calculus of primaries, the first problem which presents itself, that in which the relation of the function to its derivative is of the very simplest kind, is to investigate the nature of those functions which are equal to their derivatives. The solution of this problem leads us to classes of functions of which the exponential and the circular are cases; that is to say, the whole of the actual calculus is only the exposition of part of the simplest proposition in the theory of primaries. It is, then, not unreasonable to hope that the farther cultivation of this theory may enable us to resolve problems which have hitherto resisted all our efforts.

In arranging the parts of this theory, we may place under one general head all those cases in which the relation between the function and one of its derivatives is given, and we may call problems belonging to this head *pure problems.* Thus, when we investigate the motion of a body which is drawn toward a fixed point by an attraction depending on the distance, the relation between the function and its second derivative is given; or, if we be inquiring into the form assumed by an elastic plate, of uniform breadth, when vibrating, the relation between the ordinate and its fourth derivative is prescribed; these investigations lead to pure problems.

When two or more derivatives are combined in the relationship, we may give the name *mixed problems* to those which result. For example, the motion of a

body in a resisting medium is affected by its velocity, so that the differential equation must exhibit the second derivative of the position, in terms of that position, and of the first derivative.

And again, when the primary itself is involved, we may call the problem *adfected*. Thus, if we were to propose for investigation the motion of a body which is urged to a fixed point by an attraction proportional directly to the distance, and inversely to the time, the differential equation would take the form $-_{2t}\, x = \frac{x}{t}$, which may be called adfected. The resolution of this equation leads to a series of functions, of which the generic character is contained in the equation $x + n\,_{1t}x + t\,_{2t}x = 0$. One of these gives the form assumed by a flexible uniform chain suspended by one end and making minute oscillations.

This faint outline may serve to give an idea of the scope and classification of problems belonging to the third branch of the calculus. It makes room, as is obvious, for many physical problems which have already been resolved, and which have been regarded as belonging to the calculus of integrals. And its chapters would need to be multiplied, for the purpose of including those numerous cases in which two or more functions of the same primary are combined.

It may naturally be expected that, when broaching the subject of a new and higher branch of the calculus, I should be prepared with numerous exemplifications of its power and utility. However, I am truly in the position of a naturalist, who, having stumbled upon some unlooked-for combination of organs, some duck's bill upon the head of a quadruped, is compelled to invent a new genus, a family, even a tribe, to contain his solitary example, and who is nervously anxious to see the propriety of his extensive generalisations verified by the success of his fellow-labourers in the fields of science.

body in a resisting medium is affected by its velocity, so that the differential equation must exhibit the second derivative of the position in terms of that position and of the first derivative.

And again when the velocity itself is involved, we may call the problem [illegible]. Thus if we were to propose the investigation of the motion of a body when [illegible] to a fixed point by an attraction proportional directly to the distance, and inversely to the time, the differential equation would take the form [illegible], which may be called [illegible]. The resolution of this equation leads to a series of functions, of which the generic character is contained in the equation [illegible] $= 0$. One of these gives the form assumed by a flexible uniform chain suspended by one end and making minute oscillations.

This brief outline may serve to give an idea of the scope and classification of problems belonging to the third branch of the calculus. It must, however, be obvious, that many physical problems, which have hitherto been resolved, and which have been regarded as belonging to the calculus of integrals, would be [illegible] would need to be mentioned, for the purpose of including those numerous cases in which two or more functions of the same primary are combined.

It may naturally be expected that, when broaching the subject of a new and higher branch of the calculus, I should be prepared with numerous exemplifications of its power and utility. However, I am truly in the position of a traveller, who, having ascended upon some unknown and unexplored region of country, had a [illegible] instead of a [illegible], is compelled to arrive at a [illegible] [illegible] [illegible] [illegible] [illegible] the [illegible] of [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] of the [illegible] of [illegible].

XXXVI.—*On Functions with Recurring Derivatives.* By EDWARD SANG, Esq.

(Read 5th March 1866.)

The subject of the following paper is the first proposition in the Calculus of Primaries. The business of that calculus is to discover the relation between the primary variable and its function, when the relation subsisting between the function and its derivative is known. The simplest relationship between two variable quantities is *proportionality* when they are heterogeneous, or equality, when they are of one kind; and the case of proportionality can always, by a change in the unit of measure, be brought to an equality of the representative numbers; so that our first proposition becomes this: "To investigate the nature of those functions which reappear among their own derivatives." Since this reappearance must necessarily be periodical, I shall use the name *Functions with Recurring Derivatives.*

My attention was first drawn to functions of this class by observing that the method of solving Algebraic Equations which I published in 1829, can readily be extended to equations into which a recurring function enters. In that publication an example is given of the solution of an equation of the form $ax + b \sin x + C = 0$, and this example serves as a type for all equations composed of an algebraic and a recurring function.

1. If, after having differentiated a function several times, we come to the function itself as a differential coefficient, it is obvious that the continuation of the process must reproduce the same series of derivatives, and that the group or period must recur again and again. Hence these functions may be arranged in *orders*, according to the number of terms in the group; thus the circular functions *sine* and *cosine* belong to the fourth order.

2. Hence the sum of all the functions forming a complete period must be a recurring function of the first order.

3. When the values, corresponding to a given argument, of all the functions of a period are known, their values corresponding to another argument can be computed.

Let, for example, ϕt represent a function of the argument t, such that its third derivative is again ϕt; that is, such that ${}_{3\prime}\phi t = \phi t$; then must we have ${}_{4\prime}\phi t = {}_{1\prime}\phi t$, ${}_{5\prime}\phi t = {}_{2\prime}\phi t$, ${}_{6\prime}\phi t = {}_{3\prime}\phi t$, etc. On the supposition that t becomes $t + u$, the new function becomes, according to TAYLOR's theorem—

$$\varphi(t+u) = \varphi t + {}_{1}\varphi t\frac{u}{1} + {}_{2}\varphi t\frac{u^2}{1.2} + {}_{3}\varphi t\frac{u^3}{1.2.3} + {}_{4}\varphi t\frac{u^4}{1.2.3.4} + \text{etc.}$$

which becomes, in our present example—

$$\varphi(t+u) = \varphi t\left\{1 + \frac{u^3}{1.2.3} + \frac{u^6}{1...6} + \frac{u^9}{1....9} + \text{ etc. }\right\}$$

$$+ {}_1\varphi t\left\{\frac{u}{1} + \frac{u^4}{1...4} + \frac{u^2}{1....7} + \frac{n^{10}}{1...10} + \text{ etc. }\right\}$$

$$+ {}_2\varphi t\left\{\frac{u^2}{1.2} + \frac{u^5}{1...5} + \frac{u^8}{1....8} + \frac{u^{11}}{1...11} + \text{ etc. }\right\}$$

4. If A, B, C be the values which ϕt, ${}_1\phi t$, ${}_2\phi t$ assume when $t = o$, the above expression becomes

$$\varphi u = \text{A}\left\{1 + \frac{u^3}{1.2.3} + \text{ etc.}\right\} + \text{B}\left\{\frac{u}{1} + \frac{u^4}{1...4} + \text{ etc.}\right\} + \text{C}\left\{\frac{u^2}{1.2} + \frac{u^5}{1...5} + \text{ etc. }\right\}$$

and thus it seems that all recurring functions of the third order are compounds of multiples of the three functions—

$$1 + \frac{t^3}{1.2.3} + \frac{t^6}{1....6} + \frac{t^9}{1......9} + \text{ etc.,}$$

$$\frac{t}{1} + \frac{t^4}{1....4} + \frac{t^7}{1....7} + \frac{t^{10}}{1....10} + \text{ etc., and}$$

$$\frac{t^2}{1.2} + \frac{t^5}{1....5} + \frac{t^8}{1....8} + \frac{t^{11}}{1......11} + \text{ etc.,}$$

which may be called the fundamental functions of the third order.

Similarly, the fundamental functions of any other order, say the fourth, are—

$$1 + \frac{t^4}{1...4} + \frac{t^8}{1....8} + \frac{t^{12}}{1.....12} + \text{ etc.,}$$

$$\frac{t}{1} + \frac{t^5}{1....5} + \frac{t^9}{1....9} + \frac{t^{13}}{1.....13} + \text{ etc.,}$$

$$\frac{t^2}{1.2} + \frac{t^6}{1...6} + \frac{t^{10}}{1....10} + \frac{t^{14}}{1....14} + \text{ etc.,}$$

$$\frac{t^3}{1.2.3} + \frac{t^7}{1.....7} + \frac{t^{11}}{1....11} + \frac{t^{15}}{1......15} + \text{ etc.}$$

5. The most important of all recurring functions is that which is equal to its own first derivative; if ϕt represent such a function, we must have

$$\varphi t = {}_1\varphi t = {}_2\varphi t = {}_3\varphi t = \text{etc., and}$$

$$\varphi(t+u) = \varphi t\left\{1 + \frac{u}{1} + \frac{u^2}{1.2} + \frac{u^3}{1.2.3} + \text{ etc.}\right\}, \text{ whence}$$

$$\varphi u \quad = \text{A}\left\{1 + \frac{u}{1} + \frac{u^2}{1.2} + \frac{u^3}{1.2.3} + \text{ etc.}\right\};$$

$$\varphi t \quad = \text{A}\left\{1 + \frac{t}{u} + \frac{t^2}{1.2} + \frac{t^3}{1.2.3} + \text{ etc.}\right\};$$

and if A be unit, that is, if ϕt represent the fundamental recurring function of the first order, we have

$$\varphi(t + u) = \varphi t \,.\, \varphi u \;;$$

that is to say, the function of the sum of two arguments is the product of the functions of those arguments separately. This is merely a statement, in modern notation, of the idea which guided NEPAIR to the invention of logarithms.

Extending the above equation, we find

$$\varphi(t+u+v) = \varphi t \cdot \varphi u \cdot \varphi v,$$

which, on supposing t, u, v to be all alike, gives in general

$$\varphi(nt) = (\varphi t)^n .$$

If, according to the usual practice, we put e for the value which this function assumes when the primary is unit, that is, if we put

$$e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{ etc.},$$

we easily obtain for every integer value of n

$$\varphi n = e^n = 1 + \frac{n}{1} + \frac{n^2}{1.2} + \frac{n^3}{1.2.3} + \text{ etc.}$$

Also, if we put $nt = 1$, or $t = \frac{1}{n}$,

$$\varphi 1 = e = \varphi\left(\frac{1}{n}\right)^n, \quad \text{whence } \varphi\left(\frac{1}{n}\right) = e^{\frac{1}{n}}$$

so that the above development of e^n is true also of fractional values of n, wherefore

$$\varphi t = e^t = 1 + \frac{t}{1} + \frac{t^2}{1.2} + \frac{t^3}{1.2.3} + \text{ etc.}$$

Thus the exponential function is reached in the first step of our researches into the theory of recurring functions.

The properties of this remarkable function are too well known to be in need of any elucidation here. I may, however, indicate a rapid approximation to the n^{th} root of e, obtained by help of BROUNCKER'S continued fractions. On expanding the series for $e^{\frac{1}{n}}$ into a chain-fraction, we obtain the successive quotients $1,\ n-1,\ 1,\ 1,\ 3n-1,\ 1,\ 1,\ 5n-1,\ 1,\ 1,\ 7n-1,\ 1$, etc.; and on computing from these the series of converging fractions, we find that the 1st, 4th, 7th, 10th, etc., of them form the following progression:—

$$\begin{array}{ccccc} & 2n & 6n & 10n & 14n \\ \dfrac{1}{-1} & \dfrac{1}{1} & \dfrac{2n+1}{2n-1} & \dfrac{12n^2+6n+1}{12n^2-6n+1} & \dfrac{120n^3+60n^2+12n+1}{120n^3-60n^2+12n-1} \end{array} \text{ etc.},$$

in which the successive multipliers $2n$, $6n$, $10n$, etc., form an arithmetical progression, of which the common difference is $4n$.

6. The converse of the exponential function is the logarithmic; thus, while e^t is called the exponential function of t, t itself is called the logarithm of e^t. If we

put x for e^t our original equation $\phi t = {}_{1t}\phi t$ takes the form $x = {}_{1t}x$, and, observing that ${}_{1x}t$ is the reciprocal of ${}_{1t}x$, we have

$${}_{1x}t = x^{-1} \ .$$

The continuation of the derivation gives us ${}_{2x}t = -1.x^{-2}$; ${}_{3x}t = +1.2.x^{-3}$; ${}_{4x}t = -1.2.3.x^{-4}$; etc.; putting $\log x$ for t, and applying TAYLOR's theorem, the well-known development of a logarithm results, viz.—

$$\log(x+y) = \log x + \frac{y}{x} - \frac{1}{2}\frac{y^2}{x^2} + \frac{1}{3}\frac{y^3}{x^3} - \text{etc.}$$

7. Having said enough concerning the recurring function of the first order to show its place in the Theory of Primaries, I proceed to consider those functions which reappear as their own second derivatives.

According to what has already been explained, the fundamental functions of this order are—

$$\left.\begin{array}{l} 1 + \dfrac{t2}{1.2} + \dfrac{t4}{1...4} + \dfrac{t6}{1....6} + \text{etc.} \\[2ex] \dfrac{t}{1} + \dfrac{t3}{1.2.3} + \dfrac{t5}{1....5} + \dfrac{t7}{1....7} + \text{etc.} \end{array}\right\}$$

These functions possess very remarkable properties; and in order to exhibit these clearly, it is convenient to give distinguishing names to the two functions.

If the values of t be represented by the abscissæ, and the corresponding values of the first of the above functions by the ordinates of a series of points, those points indicate a curve which we can show to be the catenary; the length of the curve reckoned from the point corresponding to $t = o$ being the second of the two functions. On this account, and in allusion to the construction of a chain bridge, I shall designate the former function the suspensor, the latter function the catena; that is to say, I shall put

$$\text{sus}\ t = 1 + \frac{t^2}{1.2} + \frac{t^4}{1...4} + \frac{t^6}{1....6} + \text{etc.}$$

$$\text{cat}\ t = \frac{t}{1} + \frac{t^3}{1.2.3} + \frac{t^5}{1....5} + \frac{t^7}{1.....7} + \text{etc.}$$

each being the derivative of the other.

8. The differential coefficient of the square of one of these functions is just equal to that of the square of the other function,

$$\text{or } {}_{1t}(\text{sus}\ t)^2 = 2\ \text{sus}\ t\ .\ \text{cat}\ t = {}_{1t}(\text{cat}\ t)^2$$

and, consequently, the difference between these squares must be constant; now when $t = o$ that difference is unit, wherefore for all values of t

$$\text{sus}\ t^2 - \text{cat}\ t^2 = 1\ ;$$

wherefore if sus t be the absciss and cat t the ordinate of a point, that point lies in an equilateral hyperbola, so that these functions may be called *hyperbolic functions.*

9. By following the course of reasoning given in section 3, we obtain

$$\text{sus}\,(t+u) = \text{sus}\,t\,.\,\text{sus}\,u + \text{cat}\,t\,.\,\text{cat}\,u \qquad (1)$$
$$\text{sus}\,(t-u) = \text{sus}\,t\,.\,\text{sus}\,u - \text{cat}\,t\,.\,\text{cat}\,u \qquad (2)$$
$$\text{cat}\,(t+u) = \text{cat}\,t\,.\,\text{sus}\,u + \text{sus}\,t\,.\,\text{cat}\,u \qquad (3)$$
$$\text{cat}\,(t-u) = \text{cat}\,t\,.\,\text{sus}\,u - \text{sus}\,t\,.\,\text{cat}\,u \qquad (4)$$

which are the counterparts of the four elementary formulæ of trigonometry.

10. By manipulating the four equations of the preceding section, we can obtain theorems exactly analogous to those of the angular calculus; thus, by additions and subtractions,

$$\text{sus}\,(t+u) + \text{sus}\,(t-u) = 2\,.\,\text{sus}\,t\,.\,\text{sus}\,u \qquad (5)$$
$$\text{sus}\,(t+u) - \text{sus}\,(t-u) = 2\,.\,\text{cat}\,t\,.\,\text{cat}\,u \qquad (6)$$
$$\text{cat}\,(t+u) + \text{cat}\,(t-u) = 2\,.\,\text{cat}\,t\,.\,\text{sus}\,u \qquad (7)$$
$$\text{cat}\,(t+u) - \text{cat}\,(t-u) = 2\,.\,\text{sus}\,t\,.\,\text{cat}\,u \qquad (8)$$

Hence putting $t = nu$

$$\text{sus}\,(n+1)u + \text{sus}\,(n-1)u = 2\,\text{sus}\,nu\,.\,\text{sus}\,u \qquad (9)$$
$$\text{cat}\,(n+1)u + \text{cat}\,(n-1)u = 2\,\text{cat}\,.\,nu\,.\,\text{sus}\,u \qquad (10)$$
$$\text{sus}\,(n-1)u - 2\,\text{sus}\,nu + \text{sus}\,(n+1)u = \text{sus}\,nu\,.\,2(\text{sus}\,u - 1)$$
$$\text{cat}\,(n-1)u - 2\,\text{cat}\,nu + \text{cat}\,(n+1)u = \text{cat}\,nu\,.\,2(\text{sus}\,u - 1)$$

and thus a table of the values of catenarian functions may be constructed by help of second differences, just as in the case of the trigonometrical canon; the expression $2(\text{sus}\,u - 1)$ taking the place of $2(1 - \cos u)$. The same analogy may be extended to differences of the fourth and sixth orders, and so on.

11. By putting successively $t = u$, $t = 2u$, $t = 3u$, we can form the expressions for the catenarian functions of multiple arguments; thus

$$\begin{aligned}
\text{sus}\,2u &= \text{sus}\,u^2 + \text{cat}\,u^2 \\
&= 2\,.\,\text{sus}\,u^2 - 1 \qquad = 2\,.\,\text{cat}\,u^2 + 1 \\
\text{cat}\,2u &= 2\,.\,\text{sus}\,u\,.\,\text{cat}\,u \\
\text{sus}\,3u &= 4\,\text{sus}\,u^3 - 3\,\text{sus}\,u \\
\text{cat}\,3u &= 4\,\text{cat}\,u^3 + 3\,\text{cat}\,u \\
\text{sus}\,4u &= 8\,\text{sus}\,u^4 - 8\,\text{sus}\,u^2 + 1 \\
\text{cat}\,4u &= 8\,\text{cat}\,u^4 + 8\,\text{cat}\,u^2 + 1 \\
\text{sus}\,5u &= 16\,\text{sus}\,u^5 - 20\,\text{sus}\,u^3 + 5\,\text{sus}\,u \\
\text{cat}\,5u &= 16\,\text{cat}\,u^5 + 20\,\text{cat}\,u^3 + 5\,\text{cat}\,u.
\end{aligned}$$

and so on.

12. Since the sum of the functions forming a period is a recurring function of the first order, we have

$$\text{sus}\,t + \text{cat}\,t = e^t$$
but $$\text{sus}\,t^2 - \text{cat}\,t^2 = 1$$
wherefore $$\text{sus}\,t - \text{cat}\,t = e^{-t}$$

and $\text{sus}\,t = \frac{1}{2}\left(e^t + e^{-t}\right)$; $\text{cat}\,t = \frac{1}{2}\left(e^t - e^{-t}\right)$, as

is indeed evident from the ordinary operation for separating the terms containing

the odd from those containing the even powers of the argument. Hence, in general,

$$\text{sus } nt = \frac{1}{2}\left(e^{nt} + e^{-nt}\right); \quad \text{cat } nt = \frac{1}{2}\left(e^{nt} = e^{-nt}\right).$$

13. If we raise the expression for sus t to the n^{th} power, and reply the one-half of the development upon the other half, we find

$$2^n . \text{ sus } t^n = e^{nt} + e^{-nt} + \frac{n}{1}\left(e^{(n-2)t} + e^{(2-n)t}\right) + \frac{n}{1}\frac{n-1}{2}\left(e^{(n-4)t} + e^{(4-n)t}\right) + \text{etc.}$$

wherefore

$$2^{n-1} \text{ sus } t^n = \text{sus } nt + \frac{n}{1} \text{ sus}\left(n-2\right)t + \frac{n}{1}\frac{n-1}{2} \text{ sus}\left(n-4\right)t + \text{ etc.}$$

in which expression we have to distinguish the two cases of n even and n odd.

This counterpart of a celebrated trigonometrical equation is obtained by the very same process, only in this case the operations are all real; whereas the supposition $2 \cos t = x + x^{-1}$ used in the trigonometrical investigation is notoriously contrary to fact and to possibility.

By treating the function cat t in the same way, we can obtain, according as n is odd or even, the formulæ

n odd

$$2^{n-1} \text{ cat } t^n = \text{cat } nt - \frac{n}{1} \text{ cat}\left(n-2\right)t + \frac{n}{1}\frac{n-1}{2} \text{ cat}\left(n-4\right)t - \text{ etc.}$$

n even

$$2^{n-1} \text{ cat } t^n = \text{sus } nt - \frac{n}{1} \text{ sus}\left(n-2\right)t + \frac{n}{1}\frac{n-1}{2} \text{sus}\left(n-4\right)t - \text{ etc.}$$

the last term being halved when n is even.

My intention at present is not so much to give a detailed treatise on the properties of these functions as to show their close relationship to the sine and cosine of the angular calculus. A striking example of the manner in which the catenarian supplement the office of the circular functions is seen in their application to equations of the third degree.

14. All cubic equations may be brought to the form $z^3 \pm a z + b = o$, and, by making $z = x\sqrt{\frac{4a}{3}}$, these again can be changed into

$$4x^3 \pm 3x + \sqrt{\left(\frac{27b^2}{4a^3}\right)} = o$$

On comparing this last with the four equations

$$\begin{aligned} 4x^3 - 3x - \cos 3t &= o \\ 4x^3 - 3x + \sin 3t &= o \\ 4x^3 - 3x - \text{sus } 3t &= o \\ 4x^3 + 3x - \text{cat } 3t &= o \end{aligned}$$

we observe that if $\sqrt{\left(\frac{27b^2}{4a^3}\right)}$ be equal to any one of the functions of $3t$, the value of

x must be the corresponding function of t. Now, whenever $27b^2$ is greater than $4a^3$, the first two forms fail us; in such event, we must have recourse to the third equation, and when a is positive, we must use the fourth. Thus, by help of tables of angular and catenarian functions, we can resolve all equations of the third order.

15. As another instance of this close affinity, I may cite, using the notation of LEIBNITZ, the integral $\int \frac{d\varphi}{a + b \cos \varphi}$.

When a is greater than b, we make the assumption

$$\frac{b + a \cos \varphi}{a + b \cos \varphi} = \cos \psi, \text{ which gives}$$

$$\frac{\sqrt{(a^2 - b^2)}\, d\varphi}{a + b \cos \varphi} = d\psi, \text{ so that}$$

$$\int \frac{d\varphi}{a + b \cos \varphi} = \frac{1}{\sqrt{(a^2 - b^2)}} \cos^{-1} \frac{b + a \cos \varphi}{a + b \cos \varphi}$$

but when a is less than b this method of reduction fails us; in that case we may assume

$$\frac{b + a \cos \varphi}{a + b \cos \varphi} = \text{sus}\, \psi \qquad \text{when}$$

$$\frac{\sqrt{(b^2 - a^2)}\, d\varphi}{a + b \cos \varphi} = d\psi \qquad \text{and therefore}$$

$$\int \frac{d\varphi}{a + b \cos \varphi} = \frac{1}{\sqrt{(b^2 - a^2)}} \text{sus}^{-1} \frac{b + \acute{a} \cos \varphi}{a + b \cos \varphi}.$$

Conversely for the analogous integrals, we find

$$\int \frac{d\varphi}{a + b \,\text{sus}\, \varphi} = \frac{1}{\sqrt{(a^2 - b^2)}} \text{sus}^{-1} \frac{b + a \,\text{sus}\, \varphi}{a + b \,\text{sus}\, \varphi}.$$

$$\int \frac{d\varphi}{a \times b \,\text{sus}\, \varphi} = \frac{1}{\sqrt{(b^2 - a^2)}} \cos^{-1} \frac{b + a \,\text{sus}\, \varphi}{a + b \,\text{sus}\, \varphi}.$$

to which may be added

$$\int \frac{d\varphi}{a + b \,\text{cat}\, \varphi} = \frac{-1}{\sqrt{(a^2 + b^2}} \text{cat}^{-1} \frac{b - a \,\text{cat}\, \varphi}{a + b \,\text{cat}\, \varphi}.$$

16. From these examples, it is apparent that the properties of recurring functions of the second order offer a fair field for the exertions of the analyst, inasmuch as they promise to give unity to investigations which have hitherto exhibited abrupt changes.

The inquiry into the form of an equilibrated bridge, affords a good instance of their utility in physical researches. Since the whole space between the roadway and the arch stones is, in this case, filled up, the weight must be proportional to the surface of the projection on the side of the bridge. Now, in all arches, this weight is proportional to the tangent of the inclination of the arch line; this tangent is the derivative of the vertical ordinate regarded as a function of the hori-

zontal absciss, while this ordinate itself is the derivative of the area, wherefore the area of the side elevation is proportional to its second derivative, and is therefore expressed by a recurring function of the second order; and so also must be its derivative the vertical ordinate; hence the outline of the arch must be of the nature of a catenary.

17. The compounds of these catenarian functions are not less interesting than the functions themselves. Thus, we may form compounds analogous to the secant, cosecant, tangent, and cotangent of the angular calculus, and, for want of other symbols, we may designate these by underlines; thus we may put

$$\underline{\sec}\, t = \frac{1}{\text{sus}\, t}; \quad \underline{\text{cse}}\, t = \frac{1}{\text{cat}\, t}; \qquad \underline{\tan}\, t = \frac{\text{cat}\, t}{\text{sus}\, t}; \qquad \underline{\cot}\, t\, \frac{\text{sus}\, t}{\text{cat}\, t}.$$

whence we readily find

$$\underline{\tan}\, t^2 + \underline{\sec}\, t^2 = 1; \qquad \text{cat}\, t^2 - \underline{\text{cse}}\, t^2 = 1,$$

and by differentiation

$${}_{1t}\underline{\sec}\, t = -\,\underline{\sec}\, t \,.\, \underline{\tan}\, t; \qquad {}_{1t}\underline{\text{cse}}\, t = -\,\underline{\text{cse}}\, t \,.\, \underline{\cot}\, t\,;$$

$${}_{1t}\underline{\tan}\, t = +\,\underline{\sec}\, t^2; \qquad {}_{1t}\underline{\cot}\, t = -\,\underline{\text{cse}}\, t^2\,;$$

$${}_{1}\log \text{sus}\, t = \underline{\tan}\, t; \qquad {}_{1t}\log \text{cat}\, t = \underline{\cot}\, {}^2 t\,;$$

$${}_{1t}\log \underline{\tan}\, t = \underline{\tan}\, t + \underline{\cot}\, t = \underline{2\ \text{cse}}\ 2t\,.$$

18. By taking the inverse of these functions we obtain another set of differentials; thus, on putting sus $t = x$, we have cat $t = \sqrt{(x^2 - 1)}$ and $t = \overset{-1}{\text{sus}}\, x$, so that the equation ${}_{1t}\text{sus}\, t = \text{cat}\, t$ becomes

$${}_{1x}\overset{-1}{\text{sus}}\, x = (x^2 - 1)^{-\frac{1}{2}}; \text{ similarly}$$

$${}_{1x}\overset{-1}{\text{cat}}\, x = (x^2 + 1)^{-\frac{1}{2}};$$

$${}_{1x}\overset{-1}{\underline{\sec}}\, x = \frac{-1}{x\sqrt{(1 - x^2)}};$$

$${}_{1x}\overset{-1}{\underline{\text{cse}}}\, x = \frac{-1}{x\sqrt{(1 + x^2)}};$$

$${}_{1x}\overset{-1}{\underline{\tan}}\, x = (1 - x^2)^{-1};$$

$${}_{1x}\overset{-1}{\underline{\cot}}\, x = (1 - x^2)^{-1}.$$

These inverse functions are only new forms for well-known logarithmic expressions, thus

$$\overset{-1}{\text{sus}}\, x = \log \{x + \sqrt{(x^2 - 1)}\}\,; \qquad \overset{-1}{\text{cat}}\, x = \log \{x + \sqrt{(x^2 + 1)}\}$$

$$\overset{-1}{\underline{\text{sec}}}\, x = \log \frac{1 + \sqrt{(1 - x^2)}}{x}\,; \qquad \overset{-1}{\underline{\text{cse}}}\, x = \log \frac{1 + \sqrt{(1 + x^2)}}{x}\,;$$

$$\overset{-1}{\underline{\text{tan}}}\, x = \tfrac{1}{2} \log \frac{1 + x}{1 - x}\,; \qquad \overset{-1}{\text{cot}}\, x = \tfrac{1}{2} \log \frac{x + 1}{x - 1}\,;$$

but they serve to give unity of structure to those alternate integrals which are possible or impossible in circular functions, according as x is greater or less than unit.

19. The analogy of these compound catenarian or hyperbolic functions to the trigonometrical lines, is clearly shown by their geometrical representatives—thus, having measured OA, OB each equal to unit from the two sides of a right angle, let fig. 1, OC be laid off equal to some value of sus t, and draw the ordinate

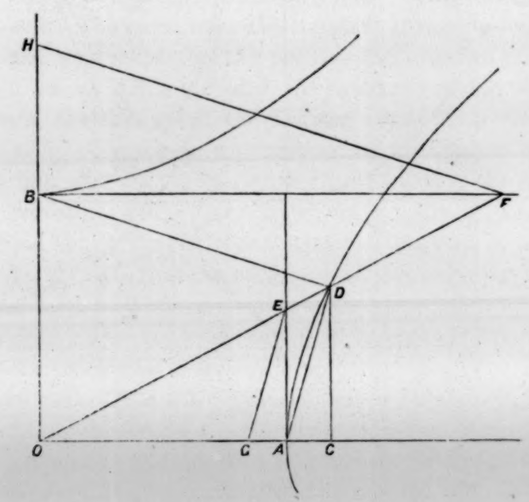


Fig. 1.

CD equal to the corresponding value of cat t, then D is a point in the equilatera hyperbola of which OA, OB are the two semi-axes.

If we draw the radius-vector OD, and suppose D to be carried to a small distance along the curve, the surface of the sector AOD will be augmented by a quantity which, in all curves, is represented by $\frac{1}{2}\{\text{OC} \,.\, \delta\text{CD} - \text{CD} \,.\, \delta\text{OC}\}$, wherefore, in the present instance, the increment of the sector AOD is $\frac{1}{2}\{\text{sus}\, t \,.\, \text{sus}\, t - \text{cat}\, t \,.\, \text{cat}\, t\}\, dt$, which, since $\text{sus}\, t^2 - \text{cat}\, t^2 = 1$, becomes $\frac{1}{2}\, dt$, therefore twice the sector AOD represents the primary variable t. Through A and B draw two per-

pendiculars meeting the radius-vector and its continuation in E and F, then AE = $\underline{\tan}\, t$, BF = $\underline{\cot}\, t$: also draw EG parallel to DA, FH parallel to DB, then OG = $\underline{\sec}\, t$, OH = $\underline{\text{cse}}\, t$.

We shall afterwards reach the functions sine and cosine when treating of recurring derivatives of the fourth order, and shall have the analogous construction shown in fig. 12. There OC is made equal to the function cos t, while CD is made equal to the sin t; but it will be shown of those functions that $\cos t^2 + \sin t^2 = 1$, wherefore D must be a point in the circumference of a circle described from O; AE is then the tangent, BF the cotangent; and, to keep up the analogy, if we draw EG parallel to DA, FH parallel to DB, OG becomes the secant, OH the cosecant of t, t being represented by the double of the sector OAD.

20. If, as in fig. 2, the absciss OT be made proportional to the primary t, and the ordinate TB to the function sus t, OA being the linear unit, the locus of the

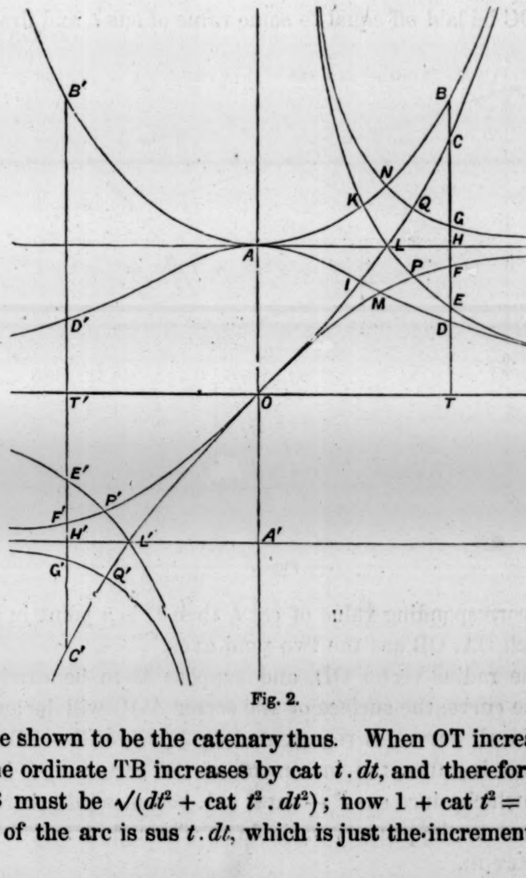


Fig. 2.

point B may be shown to be the catenary thus. When OT increases by a minute quantity dt, the ordinate TB increases by cat $t \,.\, dt$, and therefore the increment of the arc AB must be $\sqrt{(dt^2 + \text{cat } t^2 \,.\, dt^2)}$; now $1 + \text{cat } t^2 = \text{sus } t^2$, wherefore the increment of the arc is sus $t \,.\, dt$, which is just the increment of cat t, so that

the length of the arc AB must be cat t. Now the tangent of the inclination at B is also cat t, and it is the property of all arches that the tangent of the inclination is proportional to the weight reckoned from the horizontal part of the curve, and consequently the curve of which the ordinate is sus t is that which a flexible chain assumes.

Since the ordinate TB is the derivative of the area OABT regarded as a function of OT, that area must also be proportional to the function cat t; that is to say, the surface AOTB is proportional to the length of the curve AB.

21. If along the line TB we make TC proportional to the conjugate function cat t, the point C is in a curve C′OC, which we may call the companion to the catenary; this curve crosses the axis at the origin O, so that at the distance $-t$ the ordinate T′C′ appears on the opposite side; and the area OTC is proportional to HB, the excess of TB above the linear unit OA. The positive branches of these two curves approach more and more closely as t is taken of greater value.

22. By making TD a third proportional to TB and TH, we obtain the representative of the function which, for want of a better notation, we have indicated by the symbol $\underline{\text{sec}}\ t$. The curve traced by the point D rises to touch the catenary at the point A, and approaches on the positive and negative sides to the line of abscissæ. When the arc AB is unfolded, the extremity of the tangent describes, as is well known, the line called the Tractory; now if, on the surface of an oblique circular arch, a line be drawn crossing all the lines of pressure at right angles, this line indicates the proper course for the joints of the voussoirs; it is a line of double curvature, and I have shown, in a paper on the *construction of oblique arches*, read before the Society of Arts for Scotland in 1835 (*Edin. New Phil. Jour.* for April 1840), that the projection of this line upon the plane of the parapet is the *tractory*, while the projection of the same line upon a plane crossing the roadway at right angles is a modification of the curve D′AD; on that account I have called it the *companion to the tractory*.

23. If we now make TE a third proportional to TC and TH, TE becomes the representative of the function $\underline{\text{cse}}\ t$. The curve traced by E has the continuation of OA for one asymptote, and the line of abscissæ for another. For negative values of t it appears on the opposite sides of those lines, the two branches of this curve being disconnected. It crosses the companion to the catenary at t where that line crosses the parallel to OT through A.

24. Making TB : TC :: TH : TF, and TC : TB :: TH : TG we obtain TF the representative of $\underline{\text{tan}}\ t$, and TG that of $\underline{\text{cot}}\ t$. The line of *tangents* osculates and crosses the companion to the catenary at O, and has the lines AH and A′H for asymptotes; and it is remarkable that this line, to which in the abovementioned paper I have given the name *double-logarithmic*, is the projection of the same line of double curvature upon the horizontal plane. The curve of *cotangents*, traced by the points G and G′, consists of two detached branches, the positive branch

having the continuation of OA, and the parallel AH for asymptotes; the other of like dimensions approaching to the lines OA′ and A′H′.

25. I may conclude this short notice of the properties of recurring derivatives of the second order by considering the case of proportionality. Let it be proposed to investigate the nature of the function ϕt when it is proportional to its second derivative, that is when $_{2t}\phi t = c\phi t$, c being a constant multiplier.

By taking the successive derivatives of this we obtain—

$$_3\phi t = c\;.\,_1\phi t\;;\quad _4\phi t = c^2\,.\,\phi t\;;$$
$$_5\phi t = c^2\,.\,_2\phi t\;;\quad _6\phi t = c^3\,.\,\phi t\;;\quad \text{etc.},$$

whence, according to TAYLOR's theorem,

$$\phi\,(t+u) = \phi t\left\{1 + \frac{cu^2}{1.2} + \frac{c^2u^4}{1.2.3.4} + \text{etc.}\right\}$$
$$+\,c^{-\frac{1}{2}}\,{}_1\phi t\left\{\frac{c^{\frac{1}{2}}u}{1} + \frac{c^{\frac{1}{2}}u^3}{1.2.3} + \text{etc.}\right\}$$

If now we suppose that the value of ϕo is A, while that of $c^{\frac{1}{2}}{}_1\phi o$ is B, A and B being two constants, the above expression becomes

$$\phi u = \text{A}\,.\,\text{sus}\,(u\sqrt{c}) + \text{B}\,.\,\text{cat}\,(u\sqrt{c})\;,$$

which formula includes all functions which are proportional to their second derivatives.

It is here to be remarked that c cannot have a negative value; the equation,

$$_2\phi t = -\,c\,.\,\phi t\;,$$

belongs properly to recurring functions of the fourth order, under which head it will afterwards be considered.

THIRD ORDER.

The functions of the second order of recurrence are only compounds of the exponential function, and might almost have been passed over, if my sole object had been to exhibit what is novel. The functions of the third order, however, cannot be produced by compounding those of the previous orders, and we have now to touch upon ground entirely new.

26. According to what has already been explained, the fundamental recurring functions of the third order are—

$$1 + \frac{t^3}{1.2.3} + \frac{t^6}{....6} + \frac{t^9}{....9} + \text{etc.}$$
$$\frac{t}{1} + \frac{t^4}{...4} + \frac{t^7}{...7} + \frac{t^{10}}{...10} + \text{etc.}$$
$$\frac{t^2}{1.2} + \frac{t^5}{....5} + \frac{t^8}{....8} + \frac{t^{11}}{.....11} + \text{etc.}$$

For the sake of conciseness in language, it would be advantageous to have distinct

names for these. The task of finding appropriate appellations is, however, more difficult than that of finding a designation for a new asteroid or a new metal; without attempting it, I shall content myself with a simple piece of notation.

Placing the trigon $\triangle$ as the general symbol for recurring functions of the third order, we may indicate the separate cases by writing the index of the first term within it; in this way we have

$$\triangle_0 t = 1 + \frac{t^3}{1.2.3} + \frac{t^6}{\ldots 6} + \text{etc.},$$

$$\triangle_1 t = \frac{t}{1} + \frac{t^4}{\ldots 4} + \frac{t^7}{\ldots .7} + \text{etc.},$$

$$\triangle_2 t = \frac{t^2}{1.2} + \frac{t^5}{\ldots .5} + \frac{t^8}{\ldots .8} + \text{etc.},$$

and we shall afterwards use a similar notation for functions of the fourth order, these being represented by a tetragon having the appropriate numbers inscribed, thus— $\boxed{\scriptstyle 0}\,t, \boxed{\scriptstyle 1}\,t, \boxed{\scriptstyle 2}\,t, \boxed{\scriptstyle 3}\,t$.

27. Every recurring function of the third order may be represented by the formula

$$A\triangle_0 t + B\triangle_1 t + C\triangle_2 t$$

in which A , B , C are constant multipliers, which may be positive, zero, or negative. In general, if ϕt represent such a compound function, ${}_1\phi t$ and ${}_2\phi t$ being its derivatives, we have

$$\phi(t+u) = \phi t.\triangle_0 u + {}_1\phi t.\triangle_1 u + {}_2\phi t.\triangle_2 u ,$$

and, in the case of the fundamental functions themselves,

$$\triangle_0(t+u) = \triangle_0 t.\triangle_0 u + \triangle_2 t.\triangle_1 u + \triangle_1 t.\triangle_2 u , \quad (1)$$

$$\triangle_1(t+u) = \triangle_1 t.\triangle_0 u + \triangle_0 t.\triangle_1 u + \triangle_2 t.\triangle_2 u , \quad (2)$$

$$\triangle_2(t+u) = \triangle_2 t.\triangle_0 u + \triangle_1 t.\triangle_1 u + \triangle_0 t.\triangle_2 u ; \quad (3)$$

these three equations give by addition

$$\triangle_0(t+u) + \triangle_1(t+u) + \triangle_2(t+u) = (\triangle_0 t + \triangle_1 t + \triangle_2 t)(\triangle_0 u + \triangle_1 u + \triangle_2 u) , \text{ or}$$
$$e^{t+u} = e^t.e^u ;$$

they are analogous to the values of sus $(t+u)$ and cat $(t+u)$ given in article 9; but they cannot, like those, be converted into functions of the difference $t-u$ by a change of sign.

28. The sum of the cubes of three recurring functions exceeds three times the continued product of those functions by a constant quantity.

The first derivative of the sum of the cubes

$$\triangle_0 t^3 + \triangle_1 t^3 + \triangle_2 t^3 \qquad \text{is}$$

$$3\triangle_0 t^2.\triangle_2 t + 3\triangle_0 t.\triangle_1 t^2 + 3\triangle_1 t.\triangle_2 t^2$$

which is also the first derivative of

$$3\triangle_0 t\,.\,\triangle_1 t\,.\,\triangle_2 t$$

wherefore the difference

$$\triangle_0 t^3 + \triangle_1 t^3 + \triangle_2 t^3 - 3\triangle_0 t\,.\,\triangle_1 t\,.\,\triangle_2 t$$

must be constant. This proof applies to all recurring functions of the third order; in the case of the fundamental functions we have for the value $t = o$,

$$\triangle_0 o^3 + \triangle_1 o^3 + \triangle_2 o^3 - 3\triangle_0 o\,.\,\triangle_1 o\,.\,\triangle_2 o = 1\ ,$$

and, therefore, the sum of the cubes exceeds three times the continued product of those functions by unit.

29. If we substitute $-t$ for u in the equations of article 27, we have

$$\begin{aligned}
\triangle_0 o = 1 &= \triangle_0 t\,.\,\triangle_0(-t) + \triangle_2 t\,.\,\triangle_1(-t) + \triangle_1 t\,.\,\triangle_2(-t) \\
\triangle_1 o = 0 &= \triangle_1 t\,.\,\triangle_0(-t) + \triangle_0 t\,.\,\triangle_1(-t) + \triangle_2 t\,.\,\triangle_2(-t) \\
\triangle_2 o = 0 &= \triangle_2 t\,.\,\triangle_0(-t) + \triangle_1 t\,.\,\triangle_1(-t) + \triangle_0 t\,.\,\triangle_2(-t)
\end{aligned}$$

from which we can obtain the functions of $-t$ in terms of those of $+t$.

On eliminating $\triangle_1(-t)$, $\triangle_2(-t)$ from these equations, there results this

$$\triangle_0 t^2 - \triangle_1 t\,.\,\triangle_2 t = \{\triangle_0 t^3 + \triangle_1 t^3 + \triangle_2 t^3 - 3\triangle_0 t\,.\,\triangle_1 t\,.\,\triangle_2 t\}\,.\,\triangle_0(-t)$$

wherefore, according to article 28,

$$\triangle_0(-t) = \triangle_0 t^2 - \triangle_1 t\,.\,\triangle_2 t\ ;\quad \text{and similarly,} \tag{4}$$

$$\triangle_1(-t) = \triangle_2 t^2 - \triangle_0 t\,.\,\triangle_1 t\ , \tag{5}$$

$$\triangle_2(-t) = \triangle_1 t^2 - \triangle_0 t\,.\,\triangle_2 t\ . \tag{6}$$

The sum of these three functions must be e^{-t}, now on dividing *unit* under the form

$$\triangle_0 t^3 + \triangle_1 t^3 + \triangle_2 t^3 - 3\triangle_0 t\,.\,\triangle_1 t\,.\,\triangle_2 t = 1$$

by e^t under the form $\triangle_0 t + \triangle_1 t + \triangle_2 t$ we obtain the very sum in question, viz.,

$$\triangle_0 t^2 + \triangle_1 t^2 + \triangle_2 t^2 - \triangle_0 t\,.\,\triangle_1 t - \triangle_0 t\,.\,\triangle_2 t - \triangle_1 t\,.\,\triangle_2 t = e^{-t}\ . \tag{7}$$

30. If, for the sake of conciseness, we put $\mathrm{Q}t$ (quadratics) for the sum of the squares, and $\mathrm{P}t$ for the sum of the products of these functions, that is, if

$$\begin{aligned}
\mathrm{Q}t &= \triangle_0 t^2 + \triangle_1 t^2 + \triangle_2 t^2 \\
\mathrm{P}t &= \triangle_0 t\,.\,\triangle_1 t + \triangle_0 t\,.\,\triangle_2 t + \triangle_1 t\,.\,\triangle_2 t
\end{aligned}$$

we obtain, on taking the successive derivatives,

$$\begin{aligned}
{}_1\mathrm{P}t &= \mathrm{P}t + \mathrm{Q}t & \qquad {}_1\mathrm{Q}t &= 2\mathrm{P}t \\
{}_2\mathrm{P}t &= 2\mathrm{P}t + {}_1\mathrm{P}t & \qquad {}_2\mathrm{Q}t &= 2\mathrm{Q}t + {}_1\mathrm{Q}t
\end{aligned}$$

and hence each of these new functions is such that its second derivative exceeds

the double of the function by the first derivative. Continuing the derivations we find

$$\begin{array}{ll} {}_3\mathrm{P}t = 2\mathrm{P}t + 3{}_1\mathrm{P}t & {}_3\mathrm{Q}t = 2\mathrm{Q}t + 3{}_1\mathrm{Q}t \\ {}_4\mathrm{P}t = 6\mathrm{P}t + 5{}_1\mathrm{P}t & {}_4\mathrm{Q}t = 6\mathrm{Q}t + 5{}_1\mathrm{Q}t \\ {}_5\mathrm{P}t = 10\mathrm{P}t + 11{}_1\mathrm{P}t & {}_5\mathrm{Q}t = 10\mathrm{Q}t + 11{}_1\mathrm{Q}t \\ \text{etc.} & \text{etc.} \end{array}$$

The progression of the numerical coefficients in each of these series is obvious; one term augmented by the double of the preceding gives the succeeding term of the series, so that the general formula is

$${}_n\mathrm{P}t = 2\,\frac{2^{n-1} - (-1)^{n-1}}{2 \quad -(-1)}\,\mathrm{P}t + \frac{2^n - (-1)^n}{2 \quad -(-1)}\,{}_1\mathrm{P}t$$

and similarly for the function $\mathrm{Q}t$. Hence, by TAYLOR's theorem,

$$\mathrm{P}(t+u) = 2\mathrm{P}t\left\{\frac{1}{2} + 0\frac{u}{1} + 1\frac{u^2}{1.2} + 1\frac{u^3}{1.2.3} + 3\frac{u^4}{....4} + \text{etc.}\right\}$$

$$+ {}_1\mathrm{P}t\left\{1\frac{u}{1} + 1\frac{u^2}{1.2} + 3\frac{u^3}{1.2.3} + 5\frac{u^4}{...4} + 11\frac{u^5}{....5} + \text{etc.}\right\}$$

These equations are true for the P and Q of any set of recurring functions of the third order. For those with which we have at present to do, let us put $t = o$, and in the resulting formulæ change u into t, then since

$$\mathrm{Q}t + 2\mathrm{P}t = e^{2t}, \mathrm{Q}t - \mathrm{P}t = e^{-t}, \text{ we have}$$

$$\mathrm{Q}t = \frac{1}{3}\left(e^{2t} + 2e^{-t}\right) = 1 + 0\frac{t}{1} + 2\frac{t^2}{1.2} + 2\frac{t^3}{1.2.3} + 6\frac{t^4}{...4} + 10\frac{t^5}{....5} + \text{etc.},$$

$$\mathrm{P}t = \frac{1}{3}\left(e^{2t} - e^{-t}\right) = 1\frac{t}{1} + 1\frac{t^2}{1.2} + 3\frac{t^3}{1.2.3} + 5\frac{t^4}{....4} + 11\frac{t^5}{....5} + \text{etc.}$$

31. When $t = o$, the values of the three functions are $\triangle o = 1$, $\triangle o = 0$, $\triangle o = 0$, wherefore on attributing a small increment δt to this zero, the function $\triangle o$ becomes $1 + o\delta t + o\delta t^2 + \frac{1}{6}\delta t^3$; it is in a state of conjoined maximum and minimum, and augments very slowly. The function $\triangle o$ becomes $o + 1 . \delta t + o\delta t^2 + o\delta t^3$, increasing at the same rate with the primary, and the line representing it must cross the line of abscissæ at an angle of 50^c (45°). And the function $\triangle o$ becomes $o + o\delta t + \frac{1}{2}\delta t^2 + o\delta t^3$, it is therefore in a state of minimum, the radius of curvature of the line which represents it being unit. These phases are shown in figure 3, in which OA is the linear unit.

As t continues to increase, the value of $\triangle t$ approaches and becomes equal to that of $\triangle t$, that is, the corresponding curves intersect at some point B; the value of t at that instant is a root of the equation $\triangle t - \triangle t = o$; we may denote this root by T, in other words we may suppose $\triangle \mathrm{T}$ to be equal to $\triangle \mathrm{T}$.

32. If in the equations of article 27 we put t and u each equal to T, we obtain

$$\overset{0}{\triangle}2\mathrm{T} = \overset{0}{\triangle}\mathrm{T}^2 + 2\,\overset{0}{\triangle}\mathrm{T}\,.\,\overset{2}{\triangle}\mathrm{T}$$
$$\overset{1}{\triangle}2\mathrm{T} = 2\overset{0}{\triangle}\mathrm{T}^2 + \overset{2}{\triangle}\mathrm{T}^2$$
$$\overset{2}{\triangle}2\mathrm{T} = \overset{0}{\triangle}\mathrm{T}^2 + 2\,\overset{0}{\triangle}\mathrm{T}\,.\,\overset{2}{\triangle}\mathrm{T}\,,$$

wherefore, on measuring a distance of 2T along the line of abscissæ, we reach the point at which the line of $\overset{0}{\triangle}t$ is crossed by that of $\overset{2}{\triangle}t$.

Similarly, if we put 2T for t, and T for u in the same equations, we have

$$\overset{0}{\triangle}3\mathrm{T} = 2\overset{0}{\triangle}\mathrm{T}^3 + 6\overset{0}{\triangle}\mathrm{T}^2\,.\,\overset{2}{\triangle}\mathrm{T} + \overset{2}{\triangle}\mathrm{T}^3$$
$$\overset{1}{\triangle}3\mathrm{T} = 3\overset{0}{\triangle}\mathrm{T}^3 + 3\overset{0}{\triangle}\mathrm{T}^2\,.\,\overset{2}{\triangle}\mathrm{T} + 3\overset{0}{\triangle}\mathrm{T}\,.\,\overset{2}{\triangle}\mathrm{T}^2$$
$$\overset{2}{\triangle}3\mathrm{T} = 3\overset{0}{\triangle}\mathrm{T}^3 + 3\overset{0}{\triangle}\mathrm{T}^2\,.\,\overset{2}{\triangle}\mathrm{T} + 3\overset{0}{\triangle}\mathrm{T}\,.\,\overset{2}{\triangle}\mathrm{T}^2\,,$$

showing that the functions $\overset{1}{\triangle}t$ and $\overset{2}{\triangle}t$ have again become equal to each other; wherefore the distance, measured on the line of abscissæ, from the intersection of the curves 1 and 2 at the origin to their next intersection, is 3T.

33. It may be shown that this order of intersection continues indefinitely in the following manner. Let w and x be two roots of the equation $\overset{1}{\triangle}t - \overset{2}{\triangle}t = o$, then is $w + x$ also a root of the same equation;

$$\text{for} \quad \overset{1}{\triangle}(w + x) = \overset{1}{\triangle}w\,\overset{0}{\triangle}x + \overset{0}{\triangle}w\,.\,\overset{2}{\triangle}x + \overset{2}{\triangle}w\,.\,\overset{2}{\triangle}x$$
$$\text{and} \quad \overset{2}{\triangle}(w + x) = \overset{2}{\triangle}w\,\overset{0}{\triangle}x + \overset{1}{\triangle}w\,.\,\overset{1}{\triangle}x + \overset{0}{\triangle}w\,.\,\overset{2}{\triangle}x\,,$$

but by hypotheses, $\overset{1}{\triangle}w = \overset{2}{\triangle}w$, $\overset{1}{\triangle}x = \overset{2}{\triangle}x$, wherefore the above values may be written

$$\overset{1}{\triangle}(w + x) = \overset{1}{\triangle}w\,.\,\overset{0}{\triangle}x + \overset{0}{\triangle}w\,.\,\overset{1}{\triangle}x + \overset{0}{\triangle}w\,.\,\overset{1}{\triangle}x$$
$$\overset{2}{\triangle}(w + x) = \overset{1}{\triangle}w\,.\,\overset{0}{\triangle}x + \overset{1}{\triangle}w\,.\,\overset{0}{\triangle}x + \overset{0}{\triangle}w\,.\,\overset{1}{\triangle}x\,,$$

which are identic. From this it follows that the equation $\overset{1}{\triangle}t - \overset{2}{\triangle}t = o$ is satisfied by every value of t which is a multiple of 3T.

34. The same curves, viz., those representing the functions $\overset{1}{\triangle}t$ and $\overset{2}{\triangle}t$, intersect each other on the opposite side of the origin, and on ordinates situated also at the distance 3T from each other; for, according to what has been shown in article 29,

$$\overset{1}{\triangle}(-w) = \overset{2}{\triangle}w^2 - \overset{0}{\triangle}w\,.\,\overset{1}{\triangle}w$$
$$\overset{2}{\triangle}(-w) = \overset{1}{\triangle}w^2 - \overset{0}{\triangle}w\,.\,\overset{2}{\triangle}w\,,$$

wherefore if $\overset{1}{\triangle}w$ be equal to $\overset{2}{\triangle}w$, $\overset{1}{\triangle}(-w)$ must also be equal to $\overset{2}{\triangle}(-w)$, and thus the two curves intersect each other on all ordinates placed at distances $\pm 3n\mathrm{T}$ from the origin.

35. Again, if w be such a value of t as to make $\overset{1}{\triangle}w = \overset{2}{\triangle}w$, and if T be added to w, we have $\overset{0}{\triangle}(w + \mathrm{T}) = \overset{0}{\triangle}(w + \mathrm{T})$. For

$$\overset{0}{\triangle}(w + \mathrm{T}) = \overset{0}{\triangle}w\,.\,\overset{0}{\triangle}\mathrm{T} + \overset{2}{\triangle}w\,.\,\overset{1}{\triangle}\mathrm{T} + \overset{1}{\triangle}w\,.\,\overset{2}{\triangle}\mathrm{T}$$
$$\overset{1}{\triangle}(w + \mathrm{T}) = \overset{1}{\triangle}w\,.\,\overset{0}{\triangle}\mathrm{T} + \overset{0}{\triangle}w\,.\,\overset{1}{\triangle}\mathrm{T} + \overset{2}{\triangle}w\,.\,\overset{2}{\triangle}\mathrm{T}\,;$$

but, according to hypotheses $\overset{1}{\triangle}w = \overset{2}{\triangle}w$, and (article 31) $\overset{0}{\triangle}\mathrm{T} = \overset{1}{\triangle}\mathrm{T}$, wherefore

these values become

$$\triangle_0(w + T) = \triangle_0 w \,.\, \triangle_0 T + \triangle_1 w \,.\, \triangle_0 T + \triangle_1 w \,.\, \triangle_2 T$$
$$\triangle_1(w + T) = \triangle_1 w \,.\, \triangle_0 T + \triangle_0 w \,.\, \triangle_0 T + \triangle_1 w \,.\, \triangle_2 T$$

which again are identic.

36. Lastly, the curves $\triangle_0 t$ and $\triangle_2 t$ intersect on the ordinates corresponding to the abscissæ w + 2T, for similarly

$$\triangle_0(w + 2T) = \triangle_0 w \,.\, \triangle_0 2T + \triangle_2 w \,.\, \triangle_1 2T + \triangle_1 w \,.\, \triangle_2 2T$$
$$\triangle_2(w + 2T) = \triangle_2 w \,.\, \triangle_0 2T + \triangle_1 w \,.\, \triangle_1 2T + \triangle_0 w \,.\, \triangle_2 2T \;;$$

but, according to article 32, $\triangle_0 2T = \triangle_2 2T$, wherefore

$$\triangle_0(w + 2T) = \triangle_0 w \,.\, \triangle_0 2T + \triangle_1 w \,.\, \triangle_1 2T + \triangle_1 w \,.\, \triangle_0 2T$$
$$\triangle_2(w + 2T) = \triangle_1 w \,.\, \triangle_0 2T + \triangle_1 w \,.\, \triangle_1 2T + \triangle_0 w \,.\, \triangle_0 2T$$

which also are identic.

Hence it follows that the intersections of the curves take place as under

of $\triangle_0$ and $\triangle_1$ at $(1 \pm 3n)T$
of $\triangle_0$ and $\triangle_2$ at $(2 \pm 3n)T$
of $\triangle_1$ and $\triangle_2$ at $\pm 3nT$

n being any integer number whatever.

37. This distance T, or rather its triple 3T, bears a very remarkable analogy to the π of the angular calculus; and it becomes a matter of some interest to determine its value in numbers. In order to this determination we observe that the first derivative of $\triangle_0 t - \triangle_1 t$ is $\triangle_2 t - \triangle_0 t$; the second derivative $\triangle_1 t - \triangle_2 t$; while the third derivation brings us back to the original function $\triangle_0 t - \triangle_1 t$. Hence we have the subjoined calculation—

$\triangle_1 t - \triangle_2 t$	$\triangle_2 t - \triangle_0 t$	$\triangle_0 t - \triangle_1 t$	t
0·00000 00000	−1·00000 00000	+1·00000 00000	0·
+1·00000 00000	0·00000 00000	−1·00000 00000	1·
− ·50000 00000	+ ·50000 00000	·00000 00000	
·00000 00000	− ·16666 66667	+ ·16666 66667	
+ 4166 66667	0000 00000	− 4166 66667	
− 833 33333	+ 833 33333	000 00000	
00 00000	− 138 88889	+ 138 88889	
+ 19 84127	00 00000	− 19 84127	
− 2 48016	+ 2 48016	0 00000	
00000	− 27557	+ 27557	
+ 2756	0000	− 2756	
− 251	+ 251	000	
00	− 21	+ 21	
+ 2	0	− 2	

$\triangle_1 t - \triangle_2 t$	$\triangle_2 t - \triangle_0 t$	$\triangle_0 t - \triangle_1 t$	t
+ ·53350 71951	− ·65970 01534	+ ·12619 29583	1·
+ 2523 85917	+ ·10670 14390	− ·13194 00307	·2
− 1319 40031	+ 252 38592	+ 1067 01439	
+ 71 13429	− 87 96002	+ 16 82573	
+ 84129	+ 3 55671	− 4 39800	
− 17592	+ 3365	+ 14227	
+ 474	− 586	+ 112	
+ 3	+ 14	− 17	
+ ·54626 98280	− ·55131 86090	+ ·00504 87810	1·2
+ 4 64488	+ 502 56824	− 507 21312	·0092
− 2 33318	+ 2137	+ 2 31181	
+ 709	− 716	+ 7	
+ 0	+ 2	− 2	
+ ·54629 30159	− ·54629 27843	− ·00000 02315	1·2092
+ 0	− 2315	+ 2315	− 00 04238
+ ·54629 30159	− ·54629 30158	·00000 00000	1·20919 95762

which gives for T the value 1·20919 95762. In the year 1850, being in Constantinople, and having leisure, I computed this value to twenty-five places, it is

$$T = 1{\cdot}20919\ 95761\ 56145\ 23372\ 93856\ ;$$

its relation to the number $\pi = 3{\cdot}14159$, etc., might here be pointed out, but I prefer to postpone this consideration until the number π itself arise in the course of our inquiry.

38. Having now determined all those ordinates on which the curves representing the three ternary functions $\triangle_0 t$, $\triangle_1 t$, $\triangle_2 t$ cross each other, we proceed to investigate the distances intercepted by them on those ordinates. In article 29 it has been shown that the sum of the squares of these functions exceeds the sum of their products in pairs by e^{-t}, wherefore

$$(\triangle_0 t - \triangle_1 t)^2 + (\triangle_1 t - \triangle_2 t)^2 + (\triangle_2 t - \triangle_0 t)^2 = 2e^{-t}$$

so that the three lines must keep nearer and nearer to each other as t is taken greater; and thus they soon merge so closely into one line that it is impossible to delineate them separately. This line into which they merge must evidently be that of which the ordinates are $\frac{1}{3}\, e^{t}$.

39. Since for the absciss T we have $\triangle_0 T - \triangle_1 T = 0$, we must, according to the preceding article, have $(\triangle_0 T - \triangle_2 T)^2 = e^{-T}$, or

$$\triangle_0 T - \triangle_2 T = e^{-\frac{1}{2}T}$$

and consequently, the intervals intercepted on the ordinates drawn at O, T, 2T, 3T, etc., are in continued progression, the common ratio of the progression being

$$e^{-\frac{1}{2}T} = {\cdot}54629\ 30158\ 73601\ 37743\ 46503$$

the inverse ratio being

$$e^{-\frac{1}{2}T} = 1{\cdot}83051\ 94665\ 55609\ 67148\ 01998\ .$$

40. If we take any absciss u less than T, and draw an ordinate to cut the three lines; and if beyond T at the same distance u we draw another ordinate, the distances intercepted on these ordinates, between the curves, are reciprocally proportional. For since $\triangle_0 T = \triangle_1 T$, the expressions for the functions of $T + u$ may be written

$$\triangle_0(T+u) = \triangle_0 T\{\triangle_0 u + \triangle_2 u\} + \triangle_2 T \,.\, \triangle_1 u\ ,$$
$$\triangle_1(T+u) = \triangle_0 T\{\triangle_0 u + \triangle_1 u\} + \triangle_2 T \,.\, \triangle_2 u\ ,$$
$$\triangle_2(T+u) = \triangle_0 T\{\triangle_1 u + \triangle_2 u\} + \triangle_2 T \,.\, \triangle_0 u\ ;$$

wherefore,

$$\triangle_1(T+u) - \triangle_0(T+u) = (\triangle_0 T - \triangle_2 T)\{\triangle_1 u - \triangle_2 u\}$$
$$\triangle_2(T+u) - \triangle_1(T+u) = (\triangle_0 T - \triangle_2 T)\{\triangle_2 u - \triangle_0 u\}\ ;$$

that is to say, the intervals intercepted on the ordinates between T and 2T are proportional to those intercepted on the ordinates between O and T, these latter being reduced in the ratio of $e^{-\frac{1}{2}T}$ to 1, that is, of ·54629, etc., to 1.

The same law extends for every interval between nT and $(n+1)$ T, and that on both sides of the zero-point; so that if the complete details of the intersections on the ordinates between O and T were once computed, those for every other interval could thence be easily obtained.

41. If, on the other side of the origin, we measure off Ou' equal to Ou, and draw an ordinate at u', the distances intercepted thereon are again proportional to those intercepted on the ordinate at u, only in a different order. For, according to article 29, we have

$$\triangle_0(-u) = \triangle_0 u^2 - \triangle_1 u \,.\, \triangle_2 u$$
$$\triangle_1(-u) = \triangle_2 u^2 - \triangle_0 u \,.\, \triangle_1 u$$
$$\triangle_2(-u) = \triangle_1 u^2 - \triangle_0 u \,.\, \triangle_2 u$$

and, consequently,

$$\triangle_0(-u) - \triangle_1(-u) = \{\triangle_0 u - \triangle_2 u\}\, e^{u}\ ;$$
$$\triangle_1(-u) - \triangle_2(-u) = \{\triangle_2 u - \triangle_1 u\}\, e^{u}\ .$$

Hence, if we measure the distance Tu'' backwards from T and equal to Ou, the distances intercepted on the ordinate drawn through u'' are again proportional to those intercepted at u; and thus we may expect that, for the ordinate which bisects the distance OT, the intercepted distances should be alike.

For the purpose of examining into this matter, let us assume some absciss v, such that $\triangle v - \triangle v = \triangle u - \triangle v$, and let us compute the distances intercepted on the ordinate at $2v$.

We have (article 27):

$$\triangle_{0} 2v = \triangle_{0} v^2 + 2\triangle_{1} v \,.\, \triangle_{2} v \,,$$
$$\triangle_{1} 2v = \triangle_{2} v^2 + 2\triangle_{0} v \,.\, \triangle_{1} v \,,$$
$$\triangle_{2} 2v = \triangle_{1} v^2 + 2\triangle_{0} v \,.\, \triangle_{2} v \,,$$

whence

$$\triangle_{0} 2v - \triangle_{1} 2v = (\triangle_{0} v - \triangle_{2} v)\,(\triangle_{0} u - 2\triangle_{1} v + \triangle_{2} v) = 0$$

so that the two curves $\triangle_{0}$ and $\triangle_{1}$ intersect each other on the ordinate at $2v$; in other words, v must be the half of T. From this latter consideration it follows that, in tabulating the intersections, it would be enough to carry the computations as far as to $\frac{1}{2}$T, just as in the construction of the trigonometrical canon it is sufficient to make the calculations up to half a right angle.

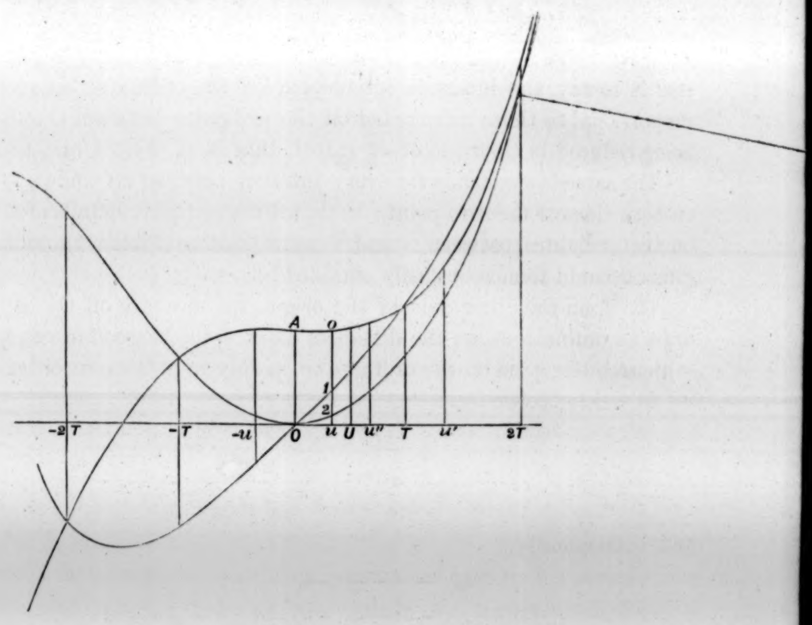


Fig. 3.

42. The relative positions of these successive intersections may be clearly exhibited by the following artifice. Let us suppose that on one of the ordinates, as that at u, figure 3, a plane is set perpendicular to the plane of the paper, and that through the points u, 2, 1, 0 lines are drawn normal to the picture; these lines being represented generally by the horizontal lines of figure 4. This arrangement being made for each successive ordinate, let an equilateral trigon ABC (fig. 4) be constructed so that its corners may be upon the

three horizontal lines drawn through 0, 1, and 2 respectively. Then, if the ordinate be supposed to move uniformly along the OT of figure 3, the side of the trigon ABC will decrease in geometrical progression, while, at the same time, it has a uniform angular motion, making one complete revolution when u moves through the distance 6T, that is, six times OT. Also the middle point of the trigon will describe a logarithmic curve, of which the ordinate is $\frac{1}{3}e^{t}$.

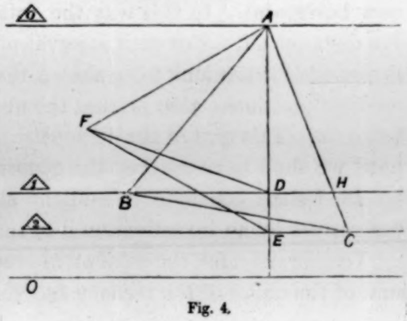


Fig. 4.

Lemma.

In order to demonstrate the truth of these assertions, let us draw through A the ordinate ADE, upon AE construct the equilateral trigon AFE, and join FD; then, since AC and AE are divided similarly at H and D, the trigons ABH, HBC are similar to AFD, DFE. Now the square of FD is obviously $AE^2 - ED \,.\, DA$, that is, $DE^2 + ED \,.\, DA + DA^2$; and it can be very easily shown that $BC^2 = \frac{4}{3} FD^2$, wherefore

$$BC^2 = \tfrac{4}{3}\{DE^2 + ED \,.\, DA + DA^2\}$$

On putting for ED and DA their values $\triangle_1 t - \triangle_2 t$ and $\triangle_0 t - \triangle_1 t$, the above formula becomes

$$BC^2 = \frac{4}{3}\left\{ \triangle_0 t^2 + \triangle_1 t^2 + \triangle_2 t^2 - \triangle_0 t \,.\, \triangle_1 t - \triangle_0 t \,.\, \triangle_2 t - \triangle_1 t \,.\, \triangle_2 t \right\}$$

and this again, according to article 29, passes into

$$BC^2 = \frac{4}{3} e^{-t}$$

wherefore the side of the equilateral trigon is given by the formula

$$BC = \sqrt{\frac{4}{3}} \,.\, e^{-\frac{1}{2}t}$$

so that, while the ordinate moves over the distance OT, the side of the equilateral trigon is diminished in the ratio of $e^{-\frac{1}{2}T}$ to unit; that is just in the ratio which has already been found for the intervals on the ordinates.

When the plane of the equilateral trigon passes along the line OA, the base BC is horizontal; as the plane moves from O towards V the end B is raised higher than C, and when that plane passes along the ordinate at V the side AC has become upright; that is to say, the trigon has made one-twelfth part of a revolution. As the plane is moved beyond V, the turning continues, and for the

ordinate at T the trigon has made one-sixth part of a turn, the side BA being now horizontal. In this way the rotation continues at the rate of one turn for the distance 6T. For each interval of $\frac{1}{2}$T the uniformity of the angular motion is obvious; it remains to be shown that the same uniformity holds good at intermediate positions; that is, that the angle of inclination is proportional to the distance Ou. This part of the demonstration, however, may be conveniently reserved until we shall have reached the genesis of angular functions.

43. I shall conclude this slight sketch of the doctrine of ternary recurring derivatives by an investigation into the sum of their cubes.

Temporarily, for the sake of abbreviation, let us put Ct as the symbol for the sum of the cubes of the ternary functions; that is, let

$$\begin{aligned}
\mathrm{C}t &= \triangle_0 t^3 + \triangle_1 t^3 + \triangle_2 t^3 \text{ , then} \\
{}_1\mathrm{C}t &= 3\{\triangle_0 t^2 \,.\, \triangle_2 t + \triangle_1 t^2 \,.\, \triangle_0 t + \triangle_2 t^2 \,.\, \triangle_1 t\} \,, \\
{}_2\mathrm{C}t &= 9\{\triangle_0 t \,.\, \triangle_2 t^2 + \triangle_1 t \,.\, \triangle_0 t^2 + \triangle_2 t \,.\, \triangle_1 t^2\} \,, \\
{}_3\mathrm{C}t &= 9\{\triangle_2 t^3 + \triangle_0 t^3 + \triangle_1 t^3 + 6\triangle_0 t \,.\, \triangle_1 t \,.\, \triangle_2 t\} \,;
\end{aligned}$$

now it has been shown (article 28) that

$$\triangle_0 t^3 + \triangle_1 t^3 + \triangle_2 t^3 - 3\triangle_0 t \,.\, \triangle_1 t \,.\, \triangle_2 t = 1$$

wherefore ${}_3\mathrm{C}t = 27\mathrm{C}t - 18$, and consequently the subsequent derivatives follow thus:—

$$\begin{aligned}
{}_4\mathrm{C}t &= 27\ {}_1\mathrm{C}t \\
{}_5\mathrm{C}t &= 27\ {}_2\mathrm{C}t \\
{}_6\mathrm{C}t &= 27^2\ \mathrm{C}t - 27 \,.\, 18 \\
{}_7\mathrm{C}t &= 27^2\ {}_1\mathrm{C}t \\
{}_8\mathrm{C}t &= 27^2\ {}_2\mathrm{C}t \\
{}_9\mathrm{C}t &= 27^3\ \mathrm{C}t - 27^2 \,.\, 18 \text{ , and so on.}
\end{aligned}$$

From these we obtain, by TAYLOR's Theorem,

$$\mathrm{C}(t+u) = \frac{2}{3}\left\{1 - \triangle_0 3u\right\} + \mathrm{C}t \,.\, \triangle_0 3u + {}_1\mathrm{C}t \,.\, \frac{1}{3}\triangle_1 3u + {}_2\mathrm{C}t \,.\, \frac{1}{9}\triangle_2 3u \,,$$

but when $t = o$, $\mathrm{C}t = 1$, ${}_1\mathrm{C}t = 0$, ${}_2\mathrm{C}t = 0$ wherefore

$$\mathrm{C}u = \frac{2}{3} + \frac{1}{3}\triangle_0 3u$$

and thus we have the following values of the sums of the triple products,

$$\begin{aligned}
\triangle_0 t^3 + \triangle_1 t^3 + \triangle_2 t^3 &= \frac{2}{3} + \frac{1}{3}\triangle_0 3t \,, \\
\triangle_0 t^2 \,.\, \triangle_2 t + \triangle_1 t^2 \,.\, \triangle_0 t + \triangle_2 t^2 \,.\, \triangle_1 t &= \frac{1}{3}\triangle_2 3t \,, \\
\triangle_0 t \,.\, \triangle_2 t^2 + \triangle_1 t \,.\, \triangle_0 t^2 + \triangle_2 t \,.\, \triangle_1 t^2 &= \frac{1}{3}\triangle_1 3t \,, \\
\triangle_0 t \,.\, \triangle_1 t \,.\, \triangle_2 t &= -\frac{1}{9} + \frac{1}{9}\triangle_0 3t \,.
\end{aligned}$$

Fourth Order.

44. According to the principles already laid down, the fundamental functions of the Fourth Order are,

$$\boxed{0}\, t = 1 + \frac{t^4}{\ldots 4} + \frac{t^8}{\ldots.8} + \frac{t^{12}}{\ldots.12} + \text{etc.}\,,$$

$$\boxed{1}\, t = \frac{t}{1} + \frac{t^5}{\ldots.5} + \frac{t^9}{\ldots.9} + \frac{t^{13}}{\ldots..13} + \text{etc.}\,,$$

$$\boxed{2}\, t = \frac{t^2}{1.2} + \frac{t^6}{\ldots.6} + \frac{t^{10}}{\ldots.10} + \frac{t^{14}}{\ldots.14} + \text{etc.}\,,$$

$$\boxed{3}\, t = \frac{t^3}{1.2.3} + \frac{t^7}{\ldots..7} + \frac{t^{11}}{\ldots.11} + \frac{t^{15}}{\ldots.15} + \text{etc.}\,,$$

and it is clear that the values of the first and third of these are the same for $-t$ as for $+t$, while those of the second and fourth merely change their signs. Hence a table of these functions for positive values of the primary can be made to serve also for negative values.

The construction of the table can be readily accomplished thus. Having prepared and titled five columns, four for the functions and one for the primary, we place at the heads of these a set of corresponding values; in order thence to compute the values corresponding to a new state of the primary, we multiply each by the increment δt of the primary and write the product in the column belonging to the function of next higher title. These results, which form the second line, are now multiplied by $\frac{1}{2}\delta t$, the products being placed in the adjoining column; the numbers entered in the third line are multiplied by $\frac{1}{3}\delta t$; those in the fourth line by $\frac{1}{4}\delta t$, and the work is carried on until the terms become insignificant. The removals, it must be carefully observed, are to be from column $\boxed{0}\, t$ to column $\boxed{1}\, t$, from $\boxed{1}\, t$ to $\boxed{2}\, t$, from $\boxed{2}\, t$ to $\boxed{3}\, t$, and lastly from $\boxed{3}\, t$ back to $\boxed{0}\, t$. The scheme of the calculation being this—

$\boxed{0}\, t$	$\boxed{1}\, t$	$\boxed{2}\, t$	$\boxed{3}\, t$	t
$\boxed{3}\, t \,.\, \delta t$	$\boxed{0}\, t \,.\, \delta t$	$\boxed{1}\, t \,.\, \delta t$	$\boxed{2}\, t \,.\, \delta t$	$+\,\delta t$
$\frac{1}{2}\boxed{2}\, t \,.\, \delta t^2$	$\frac{1}{2}\boxed{3}\, t \,.\, \delta t^2$	$\frac{1}{2}\boxed{0}\, t \,.\, \delta t^2$	$\frac{1}{2}\boxed{1}\, t \,.\, \delta t^2$	
$\frac{1}{6}\boxed{1}\, t \,.\, \delta t^3$	$\frac{1}{6}\boxed{2}\, t \,.\, \delta t^3$	$\frac{1}{6}\boxed{3}\, t \,.\, \delta t^3$	$\frac{1}{6}\boxed{0}\, t \,.\, \delta t^3$	
etc.	etc.	etc.	etc.	
$\boxed{0}\,(t+\delta t)$	$\boxed{1}\,(t+\delta t)$	$\boxed{2}\,(t+\delta t)$	$\boxed{3}\,(t+\delta t)$	$t+\delta t$

An example of the actual calculation is subjoined. Beginning at $t = 0$, the

values of the quaternary functions are 1, 0, 0, 0. From these the values of the functions $\boxed{0}\cdot 1$; $\boxed{1}\cdot 1$;

$\boxed{0}\,t$	$\boxed{1}\,t$	$\boxed{2}\,t$	$\boxed{3}\,t$	t
1·00000 00000	0·00000 00000	0·00000 00000	0·00000 00000	0·
41667	·10000 00000	500 00000	16 66667	·1
	833	14		
1·00000 41667	0·10000 00833	0·00500 00014	0·00016 66667	0·1
1 66667	·10000 04167	1000 00083	50 00002	·1
2 50000	8333	500 00208	50 00004	
1 66667	8333	278	16 66674	
41667	4167	208	7	
0	833	83	4	
0	0	14	1	
1·00000 66668	0·20000 26666	0·02000 00888	0·00133 33359	0·2
13 33336	·10000 66667	2000 02667	200 00089	·1
10 00004	66667	500 03333	100 00133	
3 33338	33333	2222	16 66778	
41670	8333	833	56	
1	833	167	17	
0	0	14	3	
1·00033 75017	0·30002 02499	0·04500 10124	0·00450 00435	0·3

$\boxed{2}\cdot 1$ and $\boxed{3}\cdot 1$ are computed by assuming the addition $\delta t = \cdot 1$; in this first computation the numerous zeroes are omitted. From these, by making another addition $\delta t = \cdot 1$, the values of the functions of ·2, and thence, again, those of the functions of ·3 are computed.

45. The functions of the sum $t + u$ are given in terms of those of t and u separately, by the four following equations :—

$$\boxed{0}(t+u) = \boxed{0}\,t\,.\,\boxed{0}\,u + \boxed{1}\,t\,.\,\boxed{3}\,u + \boxed{2}\,t\,.\,\boxed{2}\,u + \boxed{3}\,t\,.\,\boxed{1}\,u\,, \qquad (1)$$
$$\boxed{1}(t+u) = \boxed{1}\,t\,.\,\boxed{0}\,u + \boxed{2}\,t\,.\,\boxed{3}\,u + \boxed{3}\,t\,.\,\boxed{2}\,u + \boxed{0}\,t\,.\,\boxed{1}\,u\,, \qquad (2)$$
$$\boxed{2}(t+u) = \boxed{2}\,t\,.\,\boxed{0}\,u + \boxed{3}\,t\,.\,\boxed{3}\,u + \boxed{0}\,t\;\;\boxed{2}\,u + \boxed{1}\,t\,.\,\boxed{1}\,u\,, \qquad (3)$$
$$\boxed{3}(t+u) = \boxed{3}\,t\,.\,\boxed{0}\,u + \boxed{0}\,t\,.\,\boxed{3}\,u + \boxed{1}\,t\,.\,\boxed{2}\,u + \boxed{2}\,t\,.\,\boxed{1}\,u\,; \qquad (4)$$

and similarly those of the differences $t - u$ are

$$\boxed{0}(t-u) = \boxed{0}\,t\,.\,\boxed{0}\,u - \boxed{1}\,t\,.\,\boxed{3}\,u + \boxed{2}\,t\,.\,\boxed{2}\,u - \boxed{3}\,t\,.\,\boxed{1}\,u\,, \qquad (5)$$
$$\boxed{1}(t-u) = \boxed{1}\,t\,.\,\boxed{0}\,u - \boxed{2}\,t\,.\,\boxed{3}\,u + \boxed{3}\,t\,.\,\boxed{2}\,u - \boxed{0}\,t\,.\,\boxed{1}\,u\,, \qquad (6)$$
$$\boxed{2}(t-u) = \boxed{2}\,t\,.\,\boxed{0}\,u - \boxed{3}\,t\,.\,\boxed{3}\,u + \boxed{0}\,t\,.\,\boxed{2}\,u - \boxed{1}\,t\,.\,\boxed{1}\,u\,, \qquad (7)$$
$$\boxed{3}(t-u) = \boxed{3}\,t\,.\,\boxed{0}\,u - \boxed{0}\,t\,.\,\boxed{3}\,u + \boxed{1}\,t\,.\,\boxed{2}\,u - \boxed{2}\,t\,.\,\boxed{1}\,u\,. \qquad (8)$$

These follow at once from the development of the functions of $t + u$ and $t - u$, by help of TAYLOR'S Theorem. By combining these eight equations in

various ways, we may obtain a vast number of formulæ, many of which are singularly interesting.

In order to fix the mutual relations of these functions easily in the mind, we may construct a regular tetragon, and mark, as in the margin (fig. 5), the numbers 0, 1, 2, 3, at the corners, to stand for the functions $\boxed{0}$, $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ respectively. Each one is then the derivative of the succeeding, or the primitive of the preceding, taken in the order in which they are written. The six couples which can be made among these four functions may then be represented by the six lines joining the corners of the figure, of which four are lateral, and two diagonal.

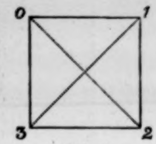
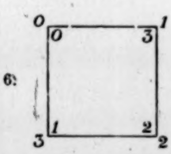
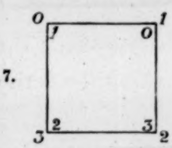


Fig. 5.

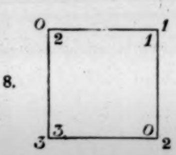
Let now a piece of paper be cut of the size of the square, and let there be written the same figures, 0, 1, 2, 3, at its four corners. If we place this paper upon figure 5, so as that the 0 may agree with the 0, we obtain the appearance shown in figure 6; and if we suppose the outer numbers to indicate functions of t, while the inner numbers indicate the corresponding functions of u, this figure 6 will at once picture the value of $\boxed{0}\,(t+u)$ in equation 1.

Fig. 6.


If we turn the inner paper until its 0 coincide with the 1 of figure 5, we obtain figure 7, which, in the same way, represents the value of $\boxed{1}\,(t+u)$; and so similarly of figures 8 and 9, which picture the values of $\boxed{2}\,(t+u)$ and $\boxed{3}\,(t+u)$.

Fig. 7.


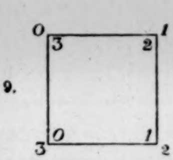
This same artifice may be applied to recurrences of higher orders, and it might have been used also for the ternary functions.

Fig. 8.


46. If we make $t = u$, equations (5) and (7) of the preceding article give,

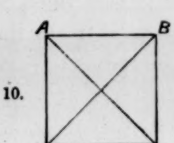
$$\boxed{0}o = 1 = \boxed{0}t^2 - 2\boxed{1}t \,.\, \boxed{3}t + \boxed{2}t^2 \qquad (9)$$

$$-\boxed{2}o = 0 = \boxed{1}t^2 - 2\boxed{0}t \,.\, \boxed{2}t + \boxed{3}t^2 \qquad (10)$$

Fig. 9.


These are two cases of a general proposition which may be exhibited thus:—

Let $\boxed{A}\,t$, $\boxed{B}\,t$, $\boxed{C}\,t$, $\boxed{D}\,t$ be the symbols of any four recurring functions, then I say that the sum of the squares of one diagonal pair differs from twice the product of the other diagonal pair by a constant quantity.

Fig. 10.


For the derivative of the sum $\boxed{A}t^2 + \boxed{C}t^2$ is $2\boxed{A}t \,.\, \boxed{D}t + 2\boxed{C}t \,.\, \boxed{B}t$; while that of $2\boxed{B}t \,.\, \boxed{D}t$ is the very same, and, consequently, for

all quaternary functions, $\boxed{A}t^2 - 2\boxed{B}t \,.\, \boxed{D}t + \boxed{C}t^2 = \text{constant},$

as also $\boxed{B}t^2 - 2\boxed{A}t \,.\, \boxed{C}t + \boxed{D}t^2 = \text{constant}.$

47. By substracting equation 10 from equation 9 we obtain

$$1 = \boxed{0}t^2 + 2\boxed{0}t \,.\, \boxed{2}t + \boxed{2}t^2 - \boxed{1}t^2 - 2\boxed{1}t \,.\, \boxed{3}t - \boxed{3}t^2, \quad \text{or}$$
$$1 = \{\boxed{0}t + \boxed{2}t\}^2 - \{\boxed{1}t + \boxed{3}t\}^2 \quad ; \quad (11)$$

now $\boxed{0}t + \boxed{2}t$ is evidently the function sus t, while $\boxed{1}t + \boxed{3}t$ is cat t, wherefore the above equation amounts to the property of catenarian functions that sus t^2 — cat $t^2 = 1$.

48. By adding together the same equations we have, on the other hand,

$$1 = \boxed{0}t^2 - 2\boxed{0}t \,.\, \boxed{2}t + \boxed{2}t^2 + \boxed{1}t^2 - 2\boxed{1}t \,.\, \boxed{3}t + \boxed{3}t^2, \quad \text{or}$$
$$1 = \{\boxed{0}t - \boxed{2}t\}^2 + \{\boxed{1}t - \boxed{3}t\}^2 \quad . \quad (12)$$

The differences $\boxed{0}t - \boxed{2}t$ and $\boxed{1}t - \boxed{3}t$ will be at once recognised as cos t and sin t respectively; and I shall at once give these titles, without, however, in the meantime attaching any signification to them, that is to say, for shortness' sake we shall put

$$\boxed{0}t - \boxed{2}t = \cos t \; ; \qquad \boxed{1}t - \boxed{3}t = \sin t \qquad (13)$$

and then equation (12) becomes

$$1 = \cos t^2 + \sin t^2 \qquad (14)$$

Taking the derivatives of equations (13) we have

$$\left.\begin{aligned} \boxed{3}t - \boxed{1}t &= {}_1\cos t = - \sin t \\ \boxed{0}t - \boxed{2}t &= {}_1\sin t = + \cos t \end{aligned}\right\} (15)$$

and thus these functions cos t, sin t are recurring functions of the Fourth Order, the order of derivation being

$$\cos t, \; - \sin t, \; - \cos t, \; + \sin t; \quad + \cos t, \text{ etc.}$$

49. If we measure distances along the line of abscissæ, corresponding to successive values of the primary variable t, and, if we set up ordinates proportional to the values of the four functions, we shall obtain four curves characteristic of these functions. For the case $t = 0$, the curve $\boxed{0}$ must pass through the point A (fig. 11), situated at the distance 0A equal to the linear unit.

Since the first derivative of $\boxed{0}t$ is there zero, the curve $\boxed{0}$ must at A be parallel to the line of abscissæ; and, since its second derivative $\boxed{2}t$ is also zero, the radius of curvature at A must be infinite, that is to say, the curve must be quite flat at A. Moreover, the third derivative $\boxed{2}t$ being also zero, and its fourth derivative, viz., itself, being positive, A must correspond to a minimum ordinate, and the curve must rise on either side of A; also the curve must be symmetrically placed on either side of the axis 0A.

The curve $\boxed{1}$ passes through the origin 0, and since the derivative of $\boxed{1}t$, viz.,

$\boxed{0}t$ is there unit, the curve $\boxed{1}$ must there be inclined at 50^{e} (45°) to the line of abscissæ. Also, since the values of $\boxed{1}t$, with their signs changed, are those of $\boxed{1}(-t)$, it follows that any straight line drawn through 0 to cut the curve $\boxed{1}$ on the one side, cuts it also at a like distance on the other side of the origin. Since the curve $\boxed{1}$ is nearly straight at 0, the second derivative of $\boxed{1}t$ being there zero, and since $\boxed{0}$ is nearly straight, and horizontal at A, the curves $\boxed{0}$ and $\boxed{1}$ must meet each other on some ordinate not far from the value $t = 1$. Now the values of the four functions when $t = 1$ are,

$$\boxed{0}1 = 1{\cdot}04169\,14703$$
$$\boxed{1}1 = 1{\cdot}00833\,60892$$
$$\boxed{2}1 = 0{\cdot}50138\,91645$$
$$\boxed{3}1 = 0{\cdot}16686\,51044\ ,$$

so that the difference $\boxed{0}1 - \boxed{1}1$ is only ·03335 53811; dividing this by its first derivative, viz., $\boxed{3}1 - \boxed{0}1$, we find the correction ·03, and thus 1·03 is nearly the value of the abscissa of the intersection of the two curves; the true value of this abscissa obtained by the subjoined computation is P = 1·03841 56373, and those of the corresponding functions are,

$$\boxed{0}P = 1{\cdot}04848\,12871$$
$$\boxed{1}P = 1{\cdot}04848\,12871$$
$$\boxed{2}P = 0{\cdot}54089\,53047$$
$$\boxed{3}P = 0{\cdot}18688\,01785\ .$$

$\boxed{0}t$	$\boxed{1}t$	$\boxed{2}t$	$\boxed{3}t$	t
1·04169 14703	1·00833 60892	0·50138 91645	0·16686 51044	1·
500 59531	3125 07441	3025 00827	1504 16749	·03
22 56251	7 50893	46 87612	45 37512	
45375	22563	7509	46876	
352	340	169	56	
0	2	2	1	
1·04692 76213	1·03966 42131	0·53210 87763	0·18236 52239	1·03
153 18679	879 41920	873 31794	446 97137	·0084
1 87728	64338	3 69356	3 66794	
1027	526	180	1034	
2	2	1	0	
1·04847 83649	1·04846 48918	0·54087 89095	0·18687 17205	1·0384
29152	1 63563	1 63561	84377	·00001 56
1	0	1	1	
1·04848 12802	1·04848 12481	0·54089 52656	0·18688 01583	1·03841 56
70	391	391	202	00373
1·04848 12872	1·04848 12872	0·54089 53047	0·18688 01785	1·03841 56373

This abscissa P, farther than that it marks the intersection of the two curves $\boxed{0}$ and $\boxed{1}$, is of no immediate interest to us.

The line $\boxed{0}$ having now crossed and passed under the line $\boxed{1}$ is next met by the line $\boxed{2}$ at a point, see figure 11, of which the abscissa is OΠ; putting Π to denote the value of this abscissa, we must have $\boxed{0}\,\Pi = \boxed{2}\,\Pi$. Now for $t = 1{\cdot}5$ we have

$$\boxed{0}\,1{\cdot}5 = 1{\cdot}21157\ 34084\ 55475\ 11794$$
$$\boxed{1}\,1{\cdot}5 = 1{\cdot}56338\ 72208\ 49435\ 96389$$
$$\boxed{2}\,1{\cdot}5 = 1{\cdot}14083\ 62067\ 87772\ 20785$$
$$\boxed{3}\,1{\cdot}5 = 0{\cdot}56589\ 22342\ 45381\ 53295$$

hence, by the subjoined process, which is exactly analogous to the preceding, we obtain

$\boxed{0}\,t$	$\boxed{1}\,t$	$\boxed{2}\,t$	$\boxed{3}\,t$	t
1·21157 34085 3961 24564 279 50487 8 93736 12121 79 2	1·56338 72208 8481 01386 138 64360 6 52178 15640 170 1	1·14083 62068 ·10943 71055 296 83549 3 23502 11413 219 2	0·56589 22342 7985 85345 383 02987 6 92616 5661 160 3	1·5 ·07
1·25407 15074 51 71221 3970 1	1·64965 05943 99 82409 2058 1	1·25327 51807 131 31219 3973 1	0·64965 09114 99 76070 5226 1	1·57 ·00079 6
1·25458 90267 2126	1·65064 90412 4100	1·25458 86999 5394	0·65064 90412 4100	1·57079 6 03268
1·25458 92393	1·65064 94512	1·25458 92393	0·65064 94512	1·57079 63268

$$\Pi = 1{\cdot}57079\ 63268$$
$$\boxed{0}\,\Pi = 1{\cdot}25458\ 92393$$
$$\boxed{1}\,\Pi = 1{\cdot}65064\ 94512$$
$$\boxed{2}\,\Pi = 1{\cdot}25458\ 92393$$
$$\boxed{3}\,\Pi = 0{\cdot}65064\ 94512$$

And here it is to be observed that the equation $\boxed{0}\,\Pi - \boxed{2}\,\Pi = 0$ is (article 49, equation 12) necessarily accompanied by this other, $\boxed{1}\,\Pi - \boxed{3}\,\Pi = 1$.

If, in equations 1, 2, 3, 4 (article 45), we suppose t and u to be each equal to the above abscissæ Π, we obtain for the quaternary functions of 2 Π the formulæ

$$\boxed{0}\,2\Pi = 2\cdot\boxed{0}\,\Pi^2 + 2\cdot\boxed{1}\,\Pi\cdot\boxed{3}\,\Pi$$
$$\boxed{1}\,2\Pi = 2\cdot\boxed{0}\,\Pi\cdot\{\boxed{1}\,\Pi + \boxed{3}\,\Pi\}$$
$$\boxed{2}\,2\Pi = 2\cdot\boxed{0}\,\Pi^2 + \boxed{1}\,\Pi^2 + \boxed{3}\,\Pi^2$$
$$\boxed{3}\,2\Pi = 2\cdot\boxed{0}\,\Pi\cdot\{\boxed{1}\,\Pi + \boxed{3}\,\Pi\} \quad ;$$

which show that the values of $\boxed{1}\,2\Pi$ and $\boxed{3}\,2\Pi$ are alike; that is to say, the two curves $\boxed{1}$ and $\boxed{3}$ cross each other on the ordinate drawn through π, if the distance 0π be made double of 0Π .

It is also seen that the value of the difference

$$\boxed{0}\,2\pi - \boxed{2}\,2\pi \quad \text{is} \quad -\boxed{1}\,\pi^2 + 2\boxed{1}\,\pi\,.\,\boxed{3}\,\pi - \boxed{3}\,\pi^2\,,$$

that is

$$-\{\boxed{1}\,\pi - \boxed{3}\,\pi\}^2 \text{ or } -1\,;$$

so that the distance intercepted on the ordinate at 2Π, between the curves $\boxed{0}$ and $\boxed{2}$ is unit.

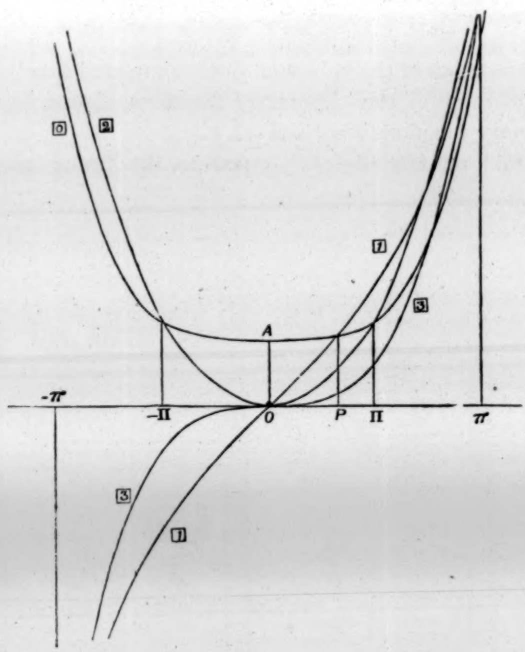


Fig. 11.

By following a train of reasoning analogous to that which was used for ternary functions, it may be shown that the intersection of the curves $\boxed{0}$ and $\boxed{2}$ are on ordinates corresponding to the abscissæ $(2n-1)\,\Pi$, while those of $\boxed{1}$ and $\boxed{3}$ are on the alternate ordinates $2n\,\Pi$, n being any integer number taken either positive

or negative; and also that on the ordinate which passes through the intersection of the one pair of lines, the distance intercepted between the other pair is the linear unit. Also, it may be proved that for the ordinates at $(n + \frac{1}{2})\Pi$ the intercepted distances are alike, each being represented by $\sqrt{\frac{1}{2}}$.

50. Leaving, for the present, the consideration of the four fundamental functions, we may give our attention to the differences $\boxed{0}t - \boxed{2}t$ and $\boxed{1}t - \boxed{3}t$, which we have already agreed to represent by the symbols cos t and sin t. By subtracting the value of $\boxed{2}(t + u)$ from that of $\boxed{0}(t + u)$, as given in article 45, we find

$$\boxed{0}(t + u) - \boxed{2}(t + u) = \{\boxed{0}t - \boxed{2}t\}\{\boxed{0}u - \boxed{2}u\} - \{\boxed{1}t - \boxed{3}t\}\{\boxed{1}u - \boxed{3}u\}$$

that is

$$\cos(t + u) = \cos t \,.\, \cos u - \sin t \,.\, \sin u\,,$$

and similarly we can obtain

$$\sin(t + u) = \sin t \,.\, \cos u + \cos t \,.\, \sin u\,,$$

so that the computation of the values of these compound functions may be made independently of those of the elementary functions, almost as we have already done for the binary functions sus t and cat t.

If, having laid off OA (fig. 12), equal to the linear unit, we make OC

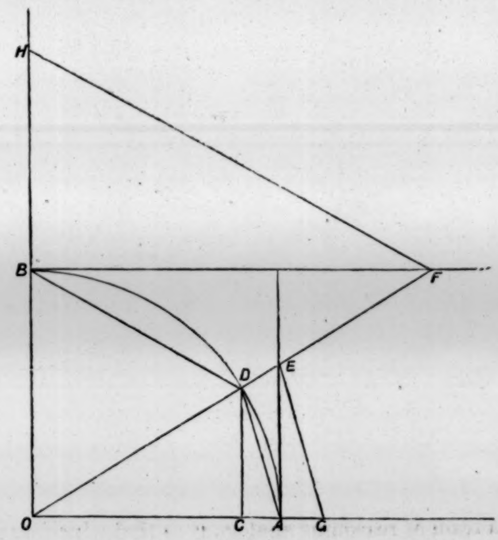


Fig. 12.

equal to the difference $\boxed{0}t - \boxed{2}t$, and set up the perpendicular CD equal to $\boxed{1}t - \boxed{3}t$; then we have $OC^2 + CD^2 = OA^2$, wherefore the point D must be in the

circumference of a circle described from O as a centre with OA as a radius. Proceeding now as in article 19, let us suppose the point D to be carried to a small distance along the curve, then the increment of the sector AOD must be represented by $\frac{1}{2}\{OC \,.\, \delta CD - CD \,.\, \delta OC\}$, that is by $\frac{1}{2}\{(\boxed{0}\,t - \boxed{2}\,t)^2 + (\boxed{1}\,t - \boxed{3}\,t)^2\}\,\delta t$ or by $\frac{1}{2}\delta t$, and consequently the double of the area AOD must represent the primary variable t. Also, since the increment of the curve is, in all cases, $\sqrt{\{\delta CD^2 + \delta OC^2\}}$, and since $\delta CD = OC \,.\, \delta t$, $\delta OC = - CD \,.\, \delta t$, it follows that the increment of the curve is just δt, so that the arc AD stands for the primary t.

Now when t is made equal to the above value of Π, the function $\boxed{0}\,t - \boxed{2}\,t$ becomes zero, so that the point C is then at 0; wherefore Π must represent the length of the quadrantal arc AB. The number π is therefore half of the well-known value $\pi = 3{\cdot}14159\,26536$ for the length of the semicircumference of a circle of which the radius is unit; and therefore, moreover, the functions $\boxed{0}\,t - \boxed{2}\,t$ and $\boxed{1}\,t - \boxed{3}\,t$, are in reality the cosine and sine of the arc t.

In this way we may imagine the whole doctrines of trigonometry as imported into our investigation, and as forming a scholium to the fourth case of the first problem in the calculus of primaries.

51. Having now arrived, in the regular course of our inquiry, at the measure of angular position, and at the circular functions, we may resume the consideration of the angular motion of the trigon ABC (fig. 4), which was left off in article 42.

If we denote the inclination of the base BC by θ, we have

$$\sin\theta = \frac{DE}{BC} = \frac{1}{2}\sqrt{3} \,.\, e^{\frac{1}{2}t}\left\{\triangle_1 t - \triangle_2 t\right\} \text{ whence}$$

$$\sin\theta^2 = \frac{3\triangle_1 t^2 - 6\triangle_1 t \,.\, \triangle_2 t + 3\triangle_2 t^2}{4\{\triangle_0 t^2 + \triangle_1 t^2 + \triangle_2 t^2 - \triangle_0 t \,.\, \triangle_1 t - \triangle_0 \,.\, \triangle_2 t - \triangle_1 t \,.\, \triangle_2 t\}}$$

$$\cos\theta^2 = \frac{4\triangle_0 t^2 - 4\triangle_0 t \,.\, \triangle_1 t - 4\triangle_0 t \,.\, \triangle_2 t + \triangle_1 t^2 + 2\triangle_1 t \,.\, \triangle_2 t + \triangle_2 t^2}{4\triangle_0 t^2 - 4\triangle_0 t \,.\, \triangle_1 t - 4\triangle_0 t \,.\, \triangle_2 t + 4\triangle_1 t^2 - 4\triangle_1 t \,.\, \triangle_2 t + 4\triangle_2 t^2}$$

and consequently

$$\cos\theta = \frac{1}{2}e^{\frac{1}{2}t}\left\{2\triangle_0 t - \triangle_1 t - \triangle_2 t\right\}$$

Taking the differential of $\sin\theta$, we have

$$\cos\theta \,.\, \delta\theta = \frac{1}{4}\sqrt{3} \,.\, e^{\frac{1}{2}t}\left\{2\triangle_0 t - \triangle_1 t - \triangle_2 t\right\}\delta t$$

$$= \frac{1}{2}\sqrt{3} \,.\, \cos\theta \,.\, \delta t \qquad \text{wherefore}$$

$$\delta\theta = \frac{1}{2}\sqrt{3} \,.\, \delta t \quad \text{and} \quad \theta = t\sqrt{\frac{3}{4}}\ ;$$

and thus, in order that the base BC may make half a turn, we must have

$$\theta = \pi \quad \text{and} \quad 3\,\mathrm{T} = \sqrt{\frac{4}{3}} \,.\, \pi \quad .$$

The value of 3T, found in article 37, is

$$3\mathrm{T} = 3{\cdot}62759\ 87283\ 68435\ 70118\ 81568$$

while

$$\pi = 3{\cdot}14159\ 26535\ 89793\ 23846\ 26434$$

and, on seeking the ratio of these by BROUNCKER's method of continued fractions, we find the quotients 1; 6, 2; 6, 2; 6, 2, the group 6, 2, being repeated ten times. It was this recurrence of the quotients, which I observed in 1850, that led me to seek for a rigid demonstration of this remarkable relation between the intervals of the intersections of the ternary curves, and those of the intersection of the quaternary ones.

52. The observation that $\sqrt{\frac{3}{4}}$ is the sine of 120°, while $-\frac{1}{2}$ is its cosine, led at once to the following unexpected generalisation.

Having assumed α any constant angle, let us put

$$\phi t = e^{t\,.\,\cos\alpha} \,.\, \sin\,(t\,.\,\sin\,\alpha)\ ,$$

then, by taking the successive derivatives, we obtain

$$_{1t}\phi t = e^{t\,.\,\cos\alpha} \,.\, \sin\,(\alpha + t\sin\alpha)$$

$$_{2t}\phi t = e^{t\,.\,\cos\alpha} \,.\, \sin\,(2\alpha + t\sin\alpha)$$

$$_{3t}\phi t = e^{t\,.\,\cos\alpha} \,.\, \sin\,(3\alpha + t\sin\alpha)$$

and in general

$$_{nt}\phi t = e^{t\,.\,\cos\alpha} \,.\, \sin\,(n\alpha + t\sin\alpha)$$

wherefore, if α be taken the nth part of the entire circumference, that is if $\alpha = \frac{2\pi}{n}$, the nth derivative of ϕt comes to be ϕt itself, and we have a series of recurring functions of the nth order. It is to be observed, however, that these are not the fundamental functions.

If we make $n = 1$, $\alpha = 2\pi$, $\cos\alpha = 1$, $\sin\alpha = 0$, ϕt becomes $e^{+t}\,.\,\sin\,(0t)$, which is quite useless. If $n = 2$, $\alpha = \pi$, $\cos\alpha = -1$, $\sin\alpha = 0$, and $\phi t = e^{-t}$ $.\,\sin\,(0t)$, which also is unavailable. But when we put $n = 3$, we have $\alpha = \frac{2}{3}\pi$ $= 120°$, $\cos\alpha = -\frac{1}{2}$, $\sin\alpha = \sqrt{\frac{3}{4}}$, and

$$\phi t = e^{-\frac{1}{2}t} \,.\, \sin\,(t\,\sqrt{\tfrac{3}{4}})$$

$$_{1t}\phi t = e^{-\frac{1}{2}t} \,.\, \sin\,(120° + t\sqrt{\tfrac{3}{4}})$$

$$_{2t}\phi = e^{-\frac{1}{2}t} \,.\, \sin\,(240° + t\,\sqrt{\tfrac{3}{4}})$$

And if we make $n = 4$, we have $a = \frac{\pi}{2}$, $\cos a = 0$, $\sin a = 1$, whence

$$\phi t = e^{ot} \sin t \qquad \text{or} \qquad \phi t = \sin t \quad .$$

These results are remarkable in this respect, that for binary functions they give the multiplier e^{-t}, which agrees with the difference between sus t and cat t; for ternary functions $e^{-\frac{1}{2}t}$, which corresponds with the diminution of the side BC of the equilateral trigon; and for quaternary functions e^{o}, in accordance with the fact that the intervals intercepted between the two curves $\boxed{0}$ and $\boxed{2}$ are the same for t as for $n\pi \pm t$, whatever may be the value of the integer number n.

The functions obtained from the formula $\phi t = e^{t\,.\,\cos a}\,.\,\sin\,(t\,.\,\sin a)$ are not fundamental, but compound functions, and do not indicate a general solution of the equation ${}_{nt}x = x$. Thus the absolutely general solution of $x = {}_{3t}x$ is

$$x = \mathrm{A}\,\triangle_{0} t + \mathrm{B}\,\triangle_{1} t + \mathrm{C}\,\triangle_{2} t$$

in which A, B, C, may be any coefficients, positive or negative; but this generality could not be obtained from the above three functions, ϕt, ${}_1\phi t$, ${}_2\phi t$.

In order to render this matter, which is of importance in physical investigations, quite clear, we may express the above ternary functions in terms of the fundamental ones, and, contrariwise, seek to deduce the fundamental functions from them. For this purpose we shall suppose that the first of them is the x of the preceding equation; that is,

$$e^{-\frac{1}{2}t}\,.\,\sin\left(t\sqrt{\tfrac{3}{4}}\right) = \mathrm{A}\,\triangle_{0} t + \mathrm{B}\,\triangle_{1} t + \mathrm{C}\,\triangle_{2} t \quad ,$$

$$e^{-\frac{1}{2}t}\,.\,\sin\left(120^\circ + t\sqrt{\tfrac{3}{4}}\right) = \mathrm{B}\,\triangle_{0} t + \mathrm{C}\,\triangle_{1} t + \mathrm{A}\,\triangle_{2} t \quad ,$$

$$e^{-\frac{1}{2}t}\,.\,\sin\left(240^\circ + t\sqrt{\tfrac{3}{4}}\right) = \mathrm{C}\,\triangle_{0} t + \mathrm{A}\,\triangle_{1} t + \mathrm{B}\,\triangle_{2} t \quad .$$

Giving in these formula to t the value zero, we obtain

$$\mathrm{A} = 0 \ , \qquad \mathrm{B} = \sin 120^\circ = \sqrt{\tfrac{3}{4}} \ ; \ \mathrm{C} = \sin 240^\circ = -\sqrt{\tfrac{3}{4}} \ ;$$

whence the three values

$$\phi t = e^{-\frac{1}{2}t}\,.\,\sin\left(t\sqrt{\tfrac{3}{4}}\right) = \sqrt{\tfrac{3}{4}}\left\{\triangle_{1} t - \triangle_{2} t\right\}$$

$${}_1\phi t = e^{-\frac{1}{2}t}\,.\,\sin\left(120^\circ + t\sqrt{\tfrac{3}{4}}\right) = \sqrt{\tfrac{3}{4}}\left\{\triangle_{0} t - \triangle_{1} t\right\}$$

$${}_2\phi t = e^{-\frac{1}{2}t}\,.\,\sin\left(240^\circ + t\sqrt{\tfrac{3}{4}}\right) = \sqrt{\tfrac{3}{4}}\left\{\triangle_{2} t - \triangle_{0} t\right\}$$

which give ϕt, ${}_1\phi t$, ${}_2\phi t$, in terms of the fundamental ternary functions. But if

we seek the values of $\triangle_{0}t$, $\triangle_{1}t$, $\triangle_{2}t$, we are met by the difficulty that the sum of the three expressions is zero, and that virtually we have only two equations whereby to determine three unknown quantities. We are thus forced to bring in the condition

$$\triangle_{0}t + \triangle_{1}t + \triangle_{2}t = e^{t} \ ,$$

by help of which we obtain the rather complex values

$$\triangle_{0}t = \frac{1}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\left\{ 2\cos\left(t\sqrt{\tfrac{3}{4}}\right) \right\} \ ,$$

$$\triangle_{1}t = \frac{1}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\left\{ -\cos\left(t\sqrt{\tfrac{3}{4}}\right) - \sqrt{3}\,.\sin\left(t\sqrt{\tfrac{3}{4}}\right) \right\} \ ,$$

$$\triangle_{2}t = \frac{1}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\left\{ -\cos\left(t\sqrt{\tfrac{3}{4}}\right) - \sqrt{3}\,.\sin\left(t\sqrt{\tfrac{3}{4}}\right) \right\} \ .$$

53. To return from this digression to the subject of quaternary functions, we observe that

$$\text{sus}\ t = \boxed{0}\,t + \boxed{2}\,t \ , \quad \text{cat}\ t = \boxed{1}\,t + \boxed{3}\,t$$

$$\cos t = \boxed{0}\,t - \boxed{2}\,t \ , \quad \sin t = \boxed{1}\,t - \boxed{3} \ ,$$

wherefore

$$\boxed{0}\,t = \frac{1}{2}\left\{ \text{sus}\ t + \cos t \right\} \ ,$$

$$\boxed{1}\,t = \frac{1}{2}\left\{ \text{cat}\ t + \sin t \right\} \ ,$$

$$\boxed{2}\,t = \frac{1}{2}\left\{ \text{sus}\ t - \cos t \right\} \ ,$$

$$\boxed{3}\,t = \frac{1}{2}\left\{ \text{cat}\ t - \sin t \right\} \ .$$

Now, all recurring functions of the fourth order are expressed by the general formula

$$\mathrm{A}\,\boxed{0}\,t + \mathrm{B}\,\boxed{1}\,t + \mathrm{C}\,\boxed{2}\,t + \mathrm{D}\,\boxed{3}\,t$$

in which A, B, C, D are any numerical coefficients; and, consequently, they may also be expressed by

$$a\,\text{sus}\ t + b\cos t + c\,\text{cat}\ t + d\sin t \ ,$$

so that the theory of quaternary recurring functions becomes a compound of the well-known doctrines of Trigonometry with the analogous and complementary doctrines of the catenarian, or, as some may prefer to call them, the hyperbolic functions. It would be easy to multiply formulæ connected with these functions, many of them interesting, on account of their relations to other researches; but as my present object is only to indicate the general features of the inquiry, I shall leave these, and proceed to apply the quaternary functions to the solution of a problem in Mechanics, which has resisted all the powers of the integral calculus.

XXXVII.—*On the Application of the Principle of Relative, or Proportional, Equality to International Organisation.* By Professor Lorimer.

(Read 18th March 1867.)

Aristotle has a saying, which he has frequently repeated and which is often quoted, to the effect that the same degree of precision is not attainable in all branches of inquiry, and that it would be just as absurd to exact demonstration from a politician or an orator, as to accept probable reasoning from a mathematician. It is a saying full of truth and acuteness. To the cultivators of ethical and political philosophy, for whom it was intended, it is invaluable both as an encouragement and a warning; and yet, in behalf of the latter more especially, I often wish that it had never been said. Proceeding from such a master, I am persuaded that it has often tempted them to rest satisfied with a degree of success far short of the limits which the nature of their subjects really imposed; whilst, on the other hand, it has afforded an apology for excluding social and political philosophy from the meditations of learned bodies like this. I do not mean that they have been formally excluded. I know that the constitution of this, and of most similar societies, has always embraced the social as well as the physical sciences. But so rarely have those of us who were occupied with the former availed ourselves of the privileges of Fellowship, that it has come to be regarded almost as a matter of admission on our part, that our subjects defy scientific treatment: that when we talk of tracing out laws of social wellbeing or progress, we use words which either have no meaning at all, or which indicate a very faint analogy between the methods which we affect to follow and those really employed in the physical sciences: and that pretty nearly all that can be done is to hand us and our subjects over to the companionship of party politicians and popular declaimers.

It is not surprising that this view should prevail, especially amongst those whose notions of the necessity of scientific precision, in other departments of study, are the strictest. It is rare to find a mathematician, or an astronomer, who does not despise politics; and I myself sympathise with their feelings to so great an extent, that it will not diminish the reverence with which I have been accustomed to regard them, nor shall I affect to view it as a mark of inhospitality, though some of the very ablest of those who listen to me should expect me to apologise for the subject which I am about to introduce to their notice this evening.

I am, however, so deeply impressed with the momentous character of the interests which centre in the social sciences, when rightly understood, that I

cannot admit the propriety of prefacing with an apology any attempt to advance them, however humble. They are the sciences of civilisation, *in the first instance*—the sciences on the more or less successful cultivation and application of which the physical sciences, which constitute so prominent a feature in the civilisation of our day, ultimately depend; for the physiologist or the entomologist, I fancy, would find himself just as much from home amongst a horde of savages as a political philosopher; and a botanist or a geologist would have very little chance of pursuing his studies in peace, even in an old society that had fallen into anarchy, or was a prey to chronic revolution.

But if the cultivators of physical science, in place of asking the cultivators of social science to apologise for their subject, were to ask them to apologise for themselves,—not for the studies which they pursue, but for the manner in which they pursue them,—the request would be difficult to put aside, however bitter might be the sarcasm which it implied. And of all the consequences of that despair of precision, to which ARISTOTLE has been but too successful in reconciling us, there is none which has brought greater reproach on our science than the want of any proper criterion of truth or falsehood. From the inability of its professors to distinguish between the difficult and the impossible,—between schemes which ought never to be relinquished, and schemes which ought never to have been entertained,—the faith of the thoughtful has been shaken, and in the thoughtless practical world without, whilst men have wasted their energies and shed their blood in wrestling with problems that were permanently insoluble, they have tamely abandoned others of the gravest import which presented no difficulties that were necessarily insuperable.

I am strongly persuaded that this reproach would never have arisen, or at all events would not have been merited, had we habituated ourselves and others to regard our subject as a *science*, in the ordinary sense of a *systematic inquiry into nature;* and not as a series of random observations, in which the contingent and the necessary, the permanent and the accidental, were hopelessly and inextricably mixed up, and from which any conclusion, or no conclusion, might equally have been deduced. Had a more absolute point of view been occupied and steadily maintained, and a severer method been rigidly adhered to, we should, long ere now, have got hold of canons of criticism which would have enabled us to judge of the merits both of existing institutions and legislative schemes, with a degree of confidence which no vague estimate of their supposed utility, past or prospective, could possibly warrant.

It is quite true that in political problems, as they present themselves in the concrete, the contingent element is so large as to prevent us from almost ever arriving at anything beyond a probable solution. The historical method, when applied exclusively, is inadequate, because the past, even if our knowledge of it were complete, does not exhaust the present, still less the future; and when we

resort to the philosophical method, we speedily become aware, that however cautious may be our application of it, fifty, or five hundred accidents may occur, which will so impede the action of necessary causes, as to render them, in a given place or for a given time, wholly inoperative. But it is a mistake to suppose that social differ altogether from physical problems in the presence of these disturbing elements, or that the separation of the accidental from the necessary is impossible in the one case more than in the other. We say, for example, that a wall will not stand if it is cracked, or that a body will fall if a vertical line through its centre of gravity falls without its base. But there are many cracked walls in this city that are very old; and the hanging tower of Pisa has stood for more than six centuries. It is nearly as old as the British Constitution, and, judging by present appearances, is, I fear, very likely to survive it. Such exceptions, of course, invalidate the rule only to the extent of showing, that cracked walls, or hanging walls, *may* stand for centuries. They reduce the presumption that such walls will tumble down within a given period to a probability; but they do not prevent us from distinguishing between the principles of physics, which *ultimately* condemn them, and the physical accidents which hold them up for a time. In the same way, though a social institution violates a principle which it cannot abrogate, or ignores a fact which it cannot alter, we must not on that account pronounce its temporary realisation to be impossible, or predict its immediate miscarriage. But, so far from believing in its permanent stability, if we know that the accidents on which it leans are transitory, and that the laws which it violates are unchangeable, if it does not right itself scientifically we may predict its practical downfall, with a confidence bordering very closely on certainty. On the other hand, if a projected institution cannot be shown to violate any such principle, or to assume as facts of nature what are not facts of nature, then there is scarcely any amount of past failure, or present difficulty, which will entitle us to exclude it from the category of attainable objects. The failures, for anything that appears, may have been accidents; and if we venture to condemn it on the strength of them, or to apply to it any of those epithets behind which ignorance and mental indolence are so eager to take shelter, we run the risk of encountering the ridicule which its ultimate success will not fail to bring down on us.

Political Methodology, viewed as a branch of applied logic, has risen in the hands of some of its recent cultivators almost to the dignity of a separate science. By eliminating impossible schemes, and thus circumscribing the sphere of political effort, it has already given evidence of its practical value for the generations that are to follow us, if not for that to which we belong. Within the State forms of government, after which the vulgar still aspire as the ideal forms of society, have been shown by its means to be permanently irreconcileable with order, and if with order, then with liberty, which is possible only through order, and so ultimately

with civilisation itself. I would gladly see the efforts of this Society directed to rendering political method more extensively available in this direction. The abiding problem of national politics I believe to be, the establishment not of the perfect State, but of *a perfect harmony between the State and the society of which it professes to be the expression;* and this problem I by no means regard as insoluble, if it were rationally and honestly dealt with. But such considerations as these would involve us in discussions which, in present circumstances, it might be difficult for us to conduct in the abstract and totally dispassionate spirit which ought always to characterise the labours of those who seek after absolute truth; and all that I shall attempt, in the meantime, is to point out to you two principles the realisation of which I believe to be impossible, and which have, nevertheless, been sought to be realised, in conjunction with most schemes of national, and I think with every scheme of international organisation which has as yet been propounded. That the latter class of schemes, from whatever cause, have miscarried in point of fact, none of you, I suppose, will have any disposition to deny; and the second part of my task will, consequently, consist of an inquiry whether, by the abandonment of the principles in question, and the substitution of their opposites, we may not hope to advance somewhat nearer to the solution of what is proclaimed on all hands to be the central problem of international jurisprudence, the establishment, viz., of a self-supporting and self-vindicating international legislature and executive.

1. The first of these principles is *finality*. In national politics this principle is exhibited in those arbitrary, and, in some States, impassible lines between classes, which science has long ago condemned, and which practical men are now everywhere engaged in obliterating. In schemes of international organisation, this principle has sought to manifest itself in the establishment of final and permanent international relations, or in the maintenance of what is technically called a *status quo*.

2. The second principle is *absolute equality of rights and obligations*. In internal politics, this principle is the basis of the form of government called Democracy. In external politics, it has exhibited itself in the custom of assigning equal votes to all the members of the family of nations not absolutely excluded from the Council Board, however widely they may differ in real power and importance.

In order that you may trace the action of these principles in international politics, I must beg you to permit me two or three sentences of historical retrospect.

It is now somewhat more than two centuries since the old dream of a Universal Empire, divinely instituted, and divinely upheld—the dream of Dante and the mediæval publicists—was abandoned, and men began to speculate on the possibility of substituting for it an European Confederation which should be

self-governing and self-supporting. The doctrine of the Balance of Power, it is true, was by no means new at the Peace of Westphalia; still less at the Treaty of Utrecht, when the name came into use. But many circumstances in the then condition of Europe lent to it an importance which it had not formerly possessed; whilst, by the institution of Permanent Embassies, which may be roughly ascribed to the same period, it was hoped that it might be worked out in practice in such a manner as to render every State that was admitted into the family of nations substantially responsible for the existence and independence of every other.

For our present purposes, then, we may assume that for two centuries, more or less, men have been striving after external organisation; and we need go no farther than the events of the last few years to convince ourselves that they have striven in vain; for never was there a period in the history of civilisation when the mutual obligations of independent communities were less recognised and acted on than at the present time. The common empire of the middle ages, never realised it is true, but never abandoned—the common church, realised beyond most human conceptions—the common language and literature which bound together the cultivated classes,—all these have been swept away, and have found, as yet, no substitutes. Dissimilar in their creeds and their institutions, their blood and their speech, the different nations of Europe, now thoroughly severed in all but their material interests, far from cherishing the sympathies of a common citizenship, scarcely exhibit those of common humanity. Armed to the teeth with the most ingenious weapons of destruction—weapons which in the end can avail only to the strongest—the full publicity and the rapid transmission of intelligence, from which so many humanising results were anticipated, seem as yet to have served scarcely any purpose but to enable rival nations to watch each other with ever-growing feelings of jealousy and distrust.

With such an experience of the fruitlessness of past effort it is not wonderful if, at times, we feel tempted to abandon all attempts at international organisation and mutual aid, and vaguely to hope that whilst separate States maintain the most absolute political independence, those indefinite influences to which we give the names of civilised opinion, moral pressure, and the like, may play amongst independent nations a part with which the most highly cultivated and most Christian communities would be very sorry to entrust them within their own borders. Yet within the State, these influences operate more potently than without it, because citizens know what fellow-citizens mean. If within the State, then, these influences become efficacious only by perfecting organisation, and thus asserting the dominion of order more unequivocally and emphatically, what reasonable hope can we entertain, that, in the great world without they will become self-acting and supply, the place of order altogether? To invoke them for such a purpose, is surely little better than to hide from ourselves, by a cloud of words, a despair to which these very words bear witness the moment that we attempt

to bring them in contact with reality, or even to fix them down to a definite sense.

But is this despair of external organisation justified by the amount of experience which these fruitless efforts to realise it as yet afford? Have we seen and done enough to warrant us in handing it over finally to the limbo of unattainable aspirations? or are there not rather points of view in which, whilst no obstacle that is insuperable in point of principle meets us, we ought to take courage from the very magnitude and difficulty of the task?

Compared with the events that make up the history of individual communities, cosmopolitan phenomena manifest themselves very slowly—so slowly, indeed, as to resemble the geological changes in the structure of the earth, rather than the mechanical changes which the works of man effect on its surface. The fact is one which all nations recognise in their ordinary speech, for we measure the progress of nations by years, or, at most, by centuries, whilst we distribute the history of mankind, on a wider scale, into Eras. But the tardiness of these greater social operations is a fact, the bearing of which on our present subject is very little regarded. We are startled, perhaps, when it occurs to us that it is only about sixty years since the Holy Roman Empire ceased to be, in name, the central institution of our own Europe. Sixty years is within the memory of man; and we should certainly look for some traces, in our present condition, of the influences of an Universal Monarchy by divine right, which had existed so recently. But when we are told that, for all practical purposes, the Holy Roman Empire perished two, or perhaps three centuries earlier—that it received its death-blow at the Reformation, and finally expired during the Thirty Years' War, leaving nothing but its shadow on the earth—we thoughtlessly hand it over to a previous stage of political existence, almost as if the sphere of its action had been on another planet. But what are two or three centuries in the history of the world? Compare them with the age of that very Empire, whether we take it from the battle of Pharsalia, or from the foundation of the Frankish Monarchy! Or, again, the life of any single Greek State was comparatively short. Sparta was regarded as a wonder of old age; and Sparta lasted, I think, only some 700 years. But if we take the Era of Greek influence, or even of the preponderance of Greek institutions, we must begin before Homer and come down to Roman times.

It is the same if we take the periods when the hegemony of the historical world was in Shemitic, and Egyptian hands.

Now if, in place of measuring the period during which the modern world has been attempting to shape itself anew, by the brief periods required for the growth and decay of national institutions, we compare it to those in which organisations of a world-wide character have been developed, we shall see reason to pause before we pronounce a confident opinion on the possibility or impossibility of so

vastan enterprise. Even if men's efforts had been well directed, their failure during two centuries of reaction against mediæval influences would not warrant their abandonment on the ground of mere lapse of time.

But have they been well directed? The history of this period, if I am not mistaken, is pregnant with lessons of warning not less emphatic than the lesson of encouragement which we derive from its comparative brevity. It tells us,—and I think the teaching of all history is to the same effect,—that the power and importance of separate communities, not only absolutely, but, what is far more important for our purpose, relatively to each other, have been continually changing, and, consequently, that what we ought to have provided for was the organisation, not of a stable body all the members of which possessed and retained definite and specific functions, but of a body in perpetual flux the members of which were changing and would continue to change their functions, their rights, and their responsibilities, relatively to each other and to the whole. But when we look into what has been really done or attempted, whether by statesmen and diplomatists for the establishment and maintenance of the Balance of Power, or by speculative politicians in their schemes for the creation of a European Confederation, or their aspirations after a Perpetual Peace, we find that the effort has invariably been to fit a final and unchangeable system to the requirements of a society which was anything but final. The very same error which dictated the search after the Perfect, or Ideal State, was thus repeated in the sphere of external politics.

To substantiate this statement by a satisfactory criticism of all, or even of any one of these schemes, would lead me beyond the limits which your time assigns to me. Those which received the sanction of diplomacy are embodied in the treaties which have followed all our great wars, and belong to general history; and I shall probably recall the general character of the other class sufficiently to your recollection when I mention the well-known names of their authors, St Pierre, Rousseau, Kant, Bentham, Cobden, and of one, the latter phases of whose much-contested policy seem to combine the practical sagacity of the statesman with the dispassionate thoughtfulness of the philosopher—I mean the Emperor Napoleon III. Should you find leisure to re-examine these various projects I believe you will find that, without a single exception, they have proposed, not only to reconstruct the map of Europe, but, when so reconstructed, to stereotype it and to guarantee, or attempt to guarantee, its permanence.

Now, it is obvious that such an interference with the natural course of events, with the ebb and flow of human fortunes, inasmuch as it assumed the possibility of controlling the strong, could have been effected only by an amount of unanimity on the part of the weak which was very unlikely to be permanent. Our utmost confidence in the doctrine of the Balance of Power could barely bring such an occurrence within the reach of possibility. But supposing it possible

that the territorial divisions of 1648 or of 1713 should have been preserved to our day, would their preservation have been just? Social existence of which political organisation without the State, as within the state, must strive to become the expression, is very far from having culminated in any direction. Not only internal development but even external aggrandisement thus frequently result from causes which are not only blameless, but in the highest degree commendable. Take, for example, a process which is constantly going on amongst States of kindred blood. The more progressive community constantly absorbs the less progressive, not by physical, but by moral and intellectual conquest. Of this we have an example in the action of Germany on Denmark, and indeed, I believe, on the whole of Scandinavia. Whatever may be the real boundary line, *for the present*, between Danes and Germans, I suppose there is not the least doubt that that line is gradually shifting *northwards;* or, in other words, that Danes are voluntarily becoming Germans, and not Germans Danes. The cause is simply the greater attractive power of the more numerous and more active body. Now, this change is one made, not by States or governments at all, but by private individuals, in the exercise of their private rights. Berlin and Vienna are more tempting fields of enterprise than Copenhagen and Stockholm: the literature of Germany is cosmopolitan, that of Scandinavia is local; and ambitious Scandinavian parents educate their sons—and ambitious Scandinavian youths educate themselves—not for a Scandinavian, but a German career. What I have said of Germany and Denmark was alleged of France and Savoy; and the real justification of the annexation was, not the pretended plebiscite, but the fact, if fact it was, that the political absorption was only a formal recognition of a moral absorption which had already deprived the lesser country of all the characteristics of a separate state. Many Frenchmen will tell you that a similar process of moral amalgamation is going on between France and Belgium. From what I remember of Savoy twenty-five years ago, I should think the allegation, as regarded it, was not altogether destitute of truth. As regards Belgium I shall offer no opinion; but this I will say, that if such facts ever do become *faits accomplis*, the sooner they become *faits de droit* the better.

In the effort, then, to struggle against inevitable change, and what at any rate *may* be legitimate progress, in the systems in question, we have got hold of the principle of *finality*, the first of the false principles which I indicated, and the presence of which alone would, in my opinion, be sufficient to account for two centuries of failure. Before we proceed to consider the possibility of its elimination from future schemes, let us turn to the other.

The *second* principle, you will remember, was *the absolute equality of all recognised States, great and small.*

As to the fact of the presence of this principle in the schemes in question, I must again presume on your historical and political knowledge. It was not the

principle which guided the negotiators at Münster in the reconstruction of the old Empire, the German Empire, as it had come to be called. The Empire, or rather the Confederation which *they* constituted, recognised and gave effect to the relative importance of the various states, by means of a complicated, but not on that account an inefficient mechanism. The good as well as the evil of feudality still clung to it, as it did to the society to which it professed to correspond. The earlier schemes of general European organisation were modelled on that of this central body, and, consequently, they exhibit the principle of equality less conspicuously than the later ones. But absolute citizen equality was the principle which the American, and, above all, the French Revolution, brought into prominence, and sought to substitute for the arbitrary and impassable barriers between class and class into which feudalism had degenerated, and which constituted the false element of finality in national organisation, social and political. As international politics came under the influence of these events, this false principle exhibited itself more and more. Since the Congress of Vienna the tendency has been to temper equality only by exclusion; and either to limit the seats at the European council board to the five great powers—the so-called "Pentarchy"—amongst whom absolute equality was the rule, or, if the lesser powers were admitted, to admit them, nominally at least, on the same footing.

But *is* the principle of absolute equality of rights and obligations between States that are unequal in importance really false in theory and unrealisable in practice? We are willing, you will say, to admit the absurdity of attempting to stereotype the map of Europe; but as to the possibility, or propriety, of recognising absolute political equality, whether within the State or without, there is, at any rate, much diversity of opinion.

Now this diversity of opinion—great as it is, and terrible as have been, and I fear may yet be, its effects—is traceable, if I am not greatly mistaken, to a defect in the popular mind, on which ARISTOTLE, with his usual perspicacity, had put his finger more than 2000 years ago. "The vulgar," he says, "do not distinguish." And in this modern Europe of ours, for nearly a century now, they have lost sight of a distinction which ARISTOTLE did them the farther favour to point out to them. The distinction to which I refer is that between *absolute*, and *relative* or *proportional, equality.*

The two are, in truth, neither more nor less than two different manifestations of the principle of justice. They differ not in themselves, but in the manner of their application, and in the subject-matter with which they deal.

Following, and giving definiteness, as usual, to PLATO's conception of what, in its origin, was probably the teaching of SOCRATES on the subject, ARISTOTLE gave to these two forms of applied justice the names of the διορθωτικὸν δίκαιον and the διανεμητικὸν δίκαιον,—names which the schoolmen and jurists rendered,

whether happily or not I shall not stop to inquire, by *justicia correctiva* or *commutativa*, and *justicia distributiva*.

The object of *diorthotic* or corrective justice, ARISTOTLE explained to be to give to each a perfectly fair, unbiassed, and, in this sense, *equal* opportunity of vindicating whatever might be due to him, whether the amount might be greater or smaller than that which was due to his neighbour. This was what we call Equality before the Law;" and justice demanded that equality, *in this sense*, should be *absolute*. There was to be no distinction whatever of rich or poor, of male or female, of old or young, of wise or foolish.

The object of *dianemetic* justice, on the other hand, was to ascertain *how much* was due to each, and to *rank* them accordingly. Here was still equality—*perfect* equality—but it was equality which was no longer absolute, but relative; it was *proportioned* to the facts which the claimants respectively established, with reference to the goods which they had acquired or inherited, or the powers and faculties which God had given them, and which their own efforts and the circumstances of their lives had developed. So far all is clear. There can be no doubt that this was what ARISTOTLE meant, and as little doubt, I think, that he was right.*

As to the application of the doctrine there is great confusion in the text, as we possess it now, and even Sir ALEXANDER GRANT has not made much of it. What it seems to indicate is, that diorthotic justice, or absolute equality, is applicable to private, and dianemetic justice, or relative equality, to public questions. So read, it excludes democracy, which rests on diorthotic, and ignores dianemetic justice altogether from the category of governments that are realisable in accordance with justice; and as this is known to have been ARISTOTLE's opinion, it is an interpretation which has satisfied most of his commentators, and I confess that it satisfied me for many years. But I am persuaded now that this could not have been ARISTOTLE's meaning; and if the text really amounts to this, it must be in consequence of some blunder or other, which may very possibly have originated with EUDEMUS. There cannot, I think, be the least doubt that *both* principles come into play in every department of jurisprudence, and are called into action in the decision of every case, from the most insignificant question of private right to the most momentous question of international policy. And the method of their action is this: the first principle, that of absolute equality, governs the

* The best minds of the middle ages preserved a perfectly clear conception of the *proportio*. "Jus," says DANTE, "est realis et personalis hominis ad hominem proportio, quæ servata servat societatem, corrupta corrumpit;" and THOMAS AQUINAS, "Materia justiciæ est exterior operatio, secundum quod ipsa, vel res qua per eam utimur, debitam proportionem habet ad alteram personam; et ideo medium justiciæ consistet in quadam proportionis æqualitate rei exterioris ad personam exteriorem," RÖDER's Natur-recht, vol. i. p. 115. It would be interesting to inquire when the vulgar conception of equality assumed the aspect of a speculative doctrine. It certainly is older than HOBBES, and is traceable as far back, at anyrate, as to the attempts of the Jesuits to found a Theocracy by levelling down all secular distinctions before the Church.

conduct of the suit, or of the investigation, whatever form it may take, and whether it be conducted for judicial or legislative purposes; the second, relative equality, governs the decision of the cause, whether that decision be pronounced in a Small Debt Court or in a Congress of Nations.

As an illustration of the mutual action of these principles in private law, take the familiar case of the distribution of a bankrupt-estate. One man has invested L.5, and another has invested L.50, in the concern. *As suitors*, the law puts them on a footing of *absolute* equality. No preference is given to Jew or to Gentile, to noble or to simple, to white skin or to black skin. So far they are dealt with diorthotically. But then the dianemetic or distributive principle comes into play; and, supposing the estate to yield 1s. per L., the one man would get 5s., and the other would get 50s. The *distribution* has reference to the objects of the suit, not to the suitors, and is wholly dianemetic. But so far is the dianemetic principle from acting alone, that it is in virtue of the diorthotic principle that it assigns 50s. to a man who may possibly be a millionaire and a scoundrel, and 5s. to a man who may be a pauper and a saint.

And just in the same way, the presence of both principles is indispensable to the decision of questions of a public nature. There, too, justice demands that the dianemetic principle shall act diorthotically. The action of the diorthotic principle in public is less obvious than in private law, because the State, in a proximate sense, is the source of the rights which it recognises; and in this sense its whole function seems to consist in *distributing*, and not in recognising rights. Still, even the State distributes, *or ought to distribute*, on a principle—on what, in the absolute sense at least, must be regarded as a foregone conclusion; and the recognition and fair application of this principle rests on, and implies the action, not of dianemetic or distributive, but of diorthotic or corrective justice. Suppose that the suffrage is claimed by a particular class of persons whose right to it has hitherto been ignored or denied. What they ask the State to do is, not to make new rights in their favour, but to recognise rights which they allege exist in their persons already. Their plea is that they are *entitled* to the suffrage on some existing *ground*, as they call it—property, education, a hearth and a chimney, or simple humanity. Whatever the ground may be, they demand that it shall be diorthotically recognised; but there the diorthotic principle stops. They don't ask the State to *give* them means on which they may ground their rights —for this would be to ask, not for *recognition*, but for *revolution;* and the same would be the case were they to ask the State to make them equal in all, or in any of these respects. The utmost limits to which the doctrines of the positive school of jurisprudence can be carried, with safety to the rights of private property, are *the recognition of the right of every man to the conditions of self-help.* Life is God's gift, and life involves freedom, and freedom involves the external conditions on which its exercise depends,—support to the impotent, instruction

to the ignorant, constraint to the criminal. So far, I think, we may go, even where these conditions are dependent on the active interposition of our fellow-men. But equality is neither a consequence of life, nor a condition of freedom. It may or may not result from free energising; and it is only where it is present as a fact that it comes within the scope of a rational representative system. It was ROUSSEAU's doctrine, that all men *ought* to be equal, and not HOBBES' assumption, that all men *are* equal, and ought *consequently* to be *recognised* as such, which brought about the Revolutions of last century, and which threatens us still. But this claim to be heard in vindication of rights which are alleged to exist, is neither more nor less than a claim to equality before the law—a claim which the constitution of this country recognises in the freedom of the press, the right to petition Parliament, to hold meetings, to form associations, and in many other ways quite as much as in the right to sue. With reference to such a claim the State, in its legislative capacity, holds, *mutatis mutandis*, precisely the position which the judge holds when the claims of two private parties are urged before him. And, in a still wider sphere, the existing States that form the commonwealth of nations assume the same position with reference to every new claimant for what, in international law, is technically known by the very significant and appropriate term of *recognition*.

If justice is to be done, then, if the principles of the science of jurisprudence—of a science which, in its minutest application, is an interpretation and an exposition of nature, are to be followed in any department whatever, there must be *proportion between the claimants* and *the things granted to them* (οἷς καὶ ἐν οἷς), in order that those who are unequal may not have equal things (μὴ ἶσοι ὀυκ ἶσα ἕξουσι), or the reverse. It is far stranger, and far more sad, than that ARISTOTLE should have failed to recognise the operation of the distributive principle in private law (if he did fail to recognise it), that the Christian world of Europe should for nearly a century have practically banished it from public law, in which ARISTOTLE held that it reigns supreme.

Such, then, being the two false principles which, since the Peace of Westphalia in 1648, have vitiated our schemes of international organisation, can we get rid of them for the future? Can we shadow forth a European or Cosmopolitan Constitution, self-sustaining and self-vindicating, which shall make provision for legitimate progress and righteous development, and for inevitable retrogression, whilst it takes cognizance of existing diversities of power?

To anything approaching to a Confederation, in the stricter sense of a single Composite State, there is, I think, the objection which exists to all Confederations, and of which we have just seen the consequences so terribly exhibited, first in America, and then in Germany. In a Confederation there are always two forces at work, a centrifugal force and a centripetal force—the tendency of the first of which is to pull it to pieces, and the tendency of the second of which is to cen-

tralise it, till it becomes a homogeneous State. A perfect and permanent balance between these forces I believe to be a practical impossibility; and for this reason, I regard all Confederations as transitional forms of government. Where absolute union must not be aimed at, even as an ultimate object, as is the case in the present instance, some modification of Federal organisation is, of course, inevitable. But the looser the bond, the less, I believe, will be the danger of its rupture; and I consequently concur in the latest opinion of KANT, whose great mind was much occupied with this great subject before it experienced the eclipse which darkened his last days—an opinion in which he was partially anticipated by GROTIUS —to the effect that it is to the creation, not of a Confederation, in any sense of the word with which we are as yet familiar, but of a Permanent Congress of Nations, or International Parliament, that we must direct our endeavours.

Such a Congress, I think, would obviate the errors I have indicated, and satisfy the great desideratum of a self-vindicating International Legislature and Executive, if it were constituted in accordance with something like the following scheme:—

1. That its meetings should be annual, taking place in the autumn between the Sessions of the various National Assemblies; and that the places of meeting should be Belgium and Switzerland alternately, or one of the Swiss Cantons, say Geneva, set apart as neutral European ground.

2. That each State should be represented by two deputies, both of whom should be present at the meetings of the Congress, but one of whom only should be entitled to speak and to vote.*

3. That each State should be entitled to vote *in proportion to its real power and importance for the time being*.

4. That in order to fix this proportion, it should be the *first* business of each Congress to ascertain the relative importance of each State, on the basis—

a. Of population.

b. Of free revenue.

c. Of exports and imports.

5. That each State be entitled to propose, and push to a vote, any question of international politics in which it might be interested.

6. That each State be bound to supply a contingent of men, or money, proportioned to the number of votes assigned to it, for the purpose of enforcing the decrees of the Congress, *by arms*, if necessary.

7. That the representatives of any State which should make war without the sanction of the Congress be excluded from its next meeting; and that the conduct of such state be judged of in the absence of its own representatives on a

* This is a proposal of BENTHAM'S, and I think there is much good sense in it, greatly as I dissent from his general principles.

written statement and oral hearing of counsel, by the representatives of the other States.

8. That all purely national questions be excluded from the deliberations of the Congress; but that the Congress itself should determine whether any question brought before it were or were not of this kind.

9. That civil wars, as opposed to rebellions, be within the jurisdiction of the Congress, the Congress itself being entitled to judge what internal commotions possess the character of civil wars.

10. That all questions brought by individual States before the Congress, be submitted to it by the representatives of such States—first, *scripto*, and then *viva voce*.

11. That a Judicial Tribunal be constituted, to the decision of which it should be competent for the Congress to remit any matter which it conceived to demand judicial determination.

12. That there should be a final appeal from this Tribunal to the Congress itself in a manner analogous to that in which the judgments of our Supreme Courts may be carried to the House of Lords.

13. That the Judges of this Court be appointed by the Congress, each State voting in proportion to its real weight, ascertained as above.

14. That the Presidents, both of the Congress itself and of the Judicial Tribunal, be appointed or re-elected at each meeting of the Congress; but that the ordinary judges of the tribunal should hold their offices *ad vitam aut culpam*.

15. That the presidents and judges, being officers of the Congress, be paid by the Congress, *and paid very highly;* but that the representatives receive no remuneration, except such as should be granted them by their respective States.

16. That the expenses of the Congress be defrayed by an international tax, to be fixed by the Congress. That the said tax be proportioned to the number of votes enjoyed for the previous year by each State, and be levied by the several States on their own inhabitants.

Many provisions of a more special kind would, of course, suggest themselves, were the scheme to assume a practical shape, but the preceding, I think, will sufficiently indicate its general character.

You will gather from the care with which I have made provision for the forcible execution of the decrees of the Congress, that I am not of the number of those who cherish very sanguine expectations of the possibility of finally and totally abolishing war. Bad as war is, we must never forget that it is a secondary evil to injustice. We must be "*first pure*, and *then* peaceable;" and it is only when an efficient substitute can be found for war, that its abolition can be rationally, *or righteously*, desired. I certainly believe that the decisions of a body which should take cognizance of the real power and importance of its various members, would have a very much better chance of being accepted

in lieu of the verdict of battle, than those of a body of which all the members voted equally. The chances would then be many, that individual States would gain no more by fighting than by voting, and to assume that, in such circumstances, they would prefer to vote, is surely to credit them with no very wonderful measure either of humanity or of wisdom. But even the pacific baton is an emblem of physical force, and without the State, as within the State, *the balance must rest on the sword.* War will cease, and ought to cease, only when the wicked cease from troubling; and I am very much of the opinion of the meditative and sarcastic Dutch innkeeper, of whom Kant relates in his Essay, that he had a churchyard painted over his door for a sign, with the superscription "Perpetual peace." On the other hand, however, I subscribe to Dr Whewell's view, that whilst war exists, the problem of its abolition is one on which all students and professors of International Law are bound perpetually to labour, because every approximation to its solution is a gain to humanity; and it is the feeling which I entertain of the very solemn character of this obligation which has emboldened me to make so serious a claim on your indulgence.

[Since the preceding paper was read to the Society, an event has occurred which must give fresh impulse to every effort to substitute diplomacy for war. In announcing the results of the Luxemburg conference to the French Legislature, on the 13th May 1867, the Marquis de Moustier said—

"The Government thinks it useful especially to point out that *for the first time*, the meeting of a Conference, instead of following a war, and confining itself to sanctioning its results, has succeeded in anticipating it, and preserving the benefits of peace. This is a precious indication of the new tendencies which prevail in the world, and over which the friends of progress and civilisation should rejoice."]

J. L.

XXXVIII.—*Some Mathematical Researches.* By H. Fox Talbot, Esq.

(Read 29th April 1867.)

I.—*On Cubic Equations.*

It is well known, that whenever the three roots of a cubic are all *real*, the solution of the equation by Cardan's rule becomes illusory. This is the more remarkable, because, *à priori*, one might have expected that the rule would only fail when the roots were imaginary. Numerous researches have been made by mathematicians on this subject; but they have not succeeded in removing this obstacle; and the only mode of finding the roots of a cubic, when all three are real, has been, by successive approximations, or the use of trigonometrical tables, or (in the case of one root being a whole number), by tentative methods and trials (which often succeed without much difficulty, when the coefficients of the equation are small numbers).

I have found, however, that there exists a certain class of cubic equations which can be solved by a process quite different from that of Cardan, and therefore not subject to any similar cause of failure. It is, moreover, exceedingly direct and simple, requiring no extraction of the cube root.

I shall suppose, for the sake of brevity, that the cubic equation wants its second term. If otherwise, the second term must be taken away by the usual rule. This being premised, the process which I speak of can be employed whereever the equation has a root of the form $a + \sqrt{b}$, where a, b, are whole numbers or rational fractions, and $\sqrt{b}$ is a Surd *in its lowest terms.*

Since $a + \sqrt{b}$ is a root, it follows that $a - \sqrt{b}$ is also a root, and the third root is $-2a$, since by hypothesis the second term of the equation is wanting. All three roots are therefore real, and all the coefficients are either whole numbers or rational fractions.

Notation.—Let $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$ be any equation. I denote the coefficients in the usual way, by p, q, r, &c. One root of the equation will be x, and I usually denote the other roots by y, z, &c.

It is well known that $p = x + y + z + \&c.$, $q = xy + yz + xz + \&c.$, $r = xyz + \&c.$, and so forth. I adopt the abbreviated notation $p = \mathrm{S}x$, $q = \mathrm{S}xy$, $r = \mathrm{S}xyz$, and so on; S standing for "the sum of," and $\mathrm{S}xy$ meaning the sum of all the combinations which can be made of two roots multiplied together; $\mathrm{S}xyz$, of three roots, and so on.

Most treatises on Algebra give easy rules by which to compute the values of

$S\,x^m$, m being any whole number, in terms of the given coefficients p, q, r, &c. (WOOD's *Algebra*, sixth edition, p. 192); and from these the values of binary compounds like $S\,x^m\,y^m$, and ternary like $S\,x^m\,y^m\,z^m$ can be computed, provided each root has the same index m.

I shall now return to cubic equations, which are the more immediate subject of this paper. The value of CARDAN's rule, when it can be applied, consists in this, that it gives an accurate result by a direct process, without guesses or tentative trials. Its inconvenience is, that the calculation is often very prolix, requiring two extractions of the cube root, although the root may be a whole number. Thus, in the example chosen by WOOD (*Algebra*, p. 172), viz., $x^3 + 6\,x - 20 = 0$, it is necessary first to extract the cube root of $10 + \sqrt{108}$, and then that of $10 - \sqrt{108}$, and to add these partial results together; which being done, their sum is found to be 2; not, however, without some rather refined arguments (see p. 132) to prove that it is *exactly* 2. Now, from the mere inspection of the equation $x^3 + 6\,x - 20 = 0$, an arithmetician would not be long in perceiving that he could solve it by supposing $x = 2$, since $8 + 12 = 20$; a process so much shorter, that it is worth while to explain why it must be disallowed. It was long ago perceived that if one of the roots of an equation was a whole number, it would necessarily be found among the divisors of the last term; and could, therefore, with more or less trouble, be found. But though this is no doubt the fact, still it cannot be admitted among the *scientific* modes of solving the equation. For, though it succeeds perfectly in an easy equation like the last, in which the last term 20 has only 2 and 5 for its prime divisors, yet in other equations the number of divisors may be so great that it would be nearly impossible to try them. For example, suppose the last term of a cubic to be 30^{100}, and that the roots are known to be whole numbers from the nature of the question which produced them. Then, since 30 is the product of the primes 2, 3, 5, it is certain that each root is a number of the form $2^a\,3^b\,5^c$. But how many trials would it not require before one of the roots, which we will suppose, for instance, to be $2^{87} \,.\, 3^{51} \,.\, 5^{19}$, would be hit upon?

For this reason, tentative processes are regarded as of doubtful value. A direct and unerring process, however long, is required, if the solution is to be regarded as a scientific one. Now, it will readily be conceded, that if ever an accurate solution is effected of equations of the 5th and higher degrees, the value of their roots will be expressed by radicals of great complication.

Nevertheless, this will not be considered to detract from the merits of the solution. The importance of such a problem is purely *theoretical;* and, therefore, provided only that the process be direct and unerring, its length has nothing whatever to do with the question; it is not intended to be used in practice, but is merely a speculation of the mind. Something similar to this is seen in the famous theorem called WILSON's Theorem, which gives a direct and certain

answer to the question, whether any given number N is a prime or not? The solution is complete, but the actual calculation, from its length, in most cases. seems impossible.

In the solution of any algebraic problem, a knowledge of arithmetic may be presupposed. As soon as the required operation is indicated, we conclude that arithmetic has performed it, by its often tedious, but steady and unerring rules; and the task of the algebraist has ended when he has shown how his problem falls within the scope of those rules.

I come now to the more immediate subject of this paper.

(1.) Let the proposed cubic be

$$x^3 + qx - r = 0$$

which will become $\quad y^3 + qy - r = 0$

and $\quad z^3 + qz - r = 0$

if the other roots are written instead of x. Adding the three equations together. we have

$$(x^3 + y^3 + z^3) + q(x + y + z) - 3r = 0$$

But $\quad x + y + z = 0 \therefore x^3 + y^3 + z^3 = 3r$

or in my notation $\quad \mathrm{S}x^3 = 3r$

(2.) Since $\quad x^3 = -qx + r$

and $\quad y^3 = -qy + r$

$\therefore$ their product $\quad x^3y^3 = q^2xy - qr(x + y) + r^2$

Whence $\quad x^3y^3 + y^3z^3 + z^3x^3 = \mathrm{S}x^3y^3 = q^2\mathrm{S}xy - 2qr\mathrm{S}x + 3r^2$

For, the process being the same with regard to each of the roots, the sum of the equations can be inferred from any one of them.

The above value of $\mathrm{S}x^3y^3$ reduces itself (since $\mathrm{S}xy = q$, $\mathrm{S}x = 0$) to

$$\mathrm{S}x^3y^3 = q^3 + 3r^2$$

(3.) There are six products of the form x^2y. It is evident that the sum of all six is equal to

$$xy(x + y) + xz(x + z) + yz(y + z)$$

But $x + y = -z$, $\therefore$ the first term becomes $-xyz$ or $-r$. Similarly for the two other terms; $\therefore$ the sum of six products like $x^2y = -3r$. But here a more important question arises. If we separate the six products into two groups of three each, *taken in order*, namely,

$$m = x^2y + y^2z + z^2x$$
$$n = y^2x + z^2y + x^2z$$

The first may be conveniently represented by $\mathrm{S}x^2y = m$, and the second by $\mathrm{S}y^2x = n$, and we are led to the inquiry, What are the values of m and n, considered separately? This problem may be solved as follows:—

We have just found that their sum $m + n$ is equal to $-3r$. Let us seek the value of their product mn.

Multiply $m = x^2y + y^2z + z^2x$ by the first term of n, which is y^2x, and we get

$$x^3y^3 + y^3 . xyz + x^2y^2z^2$$

Then multiply by the second and third terms of n, and take the sum of all the results, and we evidently shall have

$$mn = \mathrm{S}x^3y^3 + r\mathrm{S}x^3 + 3r^2$$

$\therefore$ substituting the values already found of

$$\mathrm{S}x^3 = 3r \quad \text{and} \quad \mathrm{S}x^3y^3 = q^3 + 3r^2$$

we find $\qquad mn = q^3 + 9r^2$

Now we have found $\qquad (m + n)^2 = 9r^2$

and from this subtracting $\qquad 4mn = 4q^3 + 36r^2$

we get $\qquad (m - n)^2 = -4q^3 - 27r^2$

from whence the separate values of m and n follow at once.

(4.) If the roots of a cubic, written in any order, are x, y, z, the differences of the roots, *taken in order*, are $x - y$, $y - z$, $z - x$. The other three differences, $y - x$, &c., are merely the negatives of the three first.

THEOREM.—In any cubic equation, wanting the second term, the product of the three differences of the roots, or,

$$\pm \overline{x - y} . \overline{y - z} . \overline{z - x} = \sqrt{-(4q^3 + 27r^2)}$$

For, by actual multiplication, we find

$$\begin{array}{c} z - x \\ y - z \\ \hline yz - xy - z^2 + xz \\ \\ x - y \\ \hline xyz - x^2y - xz^2 + x^2z - y^2z + xy^2 + yz^2 - xyz . \end{array}$$

Omitting the first and last terms, which destroy each other, the result is $(-x^2y - y^2z - z^2x) + (y^2x + z^2y + a^2z)$, or $-\mathrm{S}x^2y + \mathrm{S}y^2x$, or, according to our previous notation, $-m + n$. If we take the product of the other three differences of the roots, we get $m - n$.

But we have found the value of $m - n$ to be $\sqrt{-(4q^3 + 27r^2)}$; therefore the theorem is proved. Several important consequences follow. In the first place, if the roots are whole numbers, their differences are so; whence this theorem.

"If the roots of the cubic $x^3 + qx - r = 0$ are whole numbers, the quantity $4q^3 + 27r^2$ is necessarily the negative of a square." This theorem is, I believe, due to LEGENDRE. To give a few examples of it—

Example 1. Let $x^3 - 7x + 6 = 0$, the roots are $1, 2, -3$, $q = 7$, $r = -6$, $4q^3 = -1372$ and $27r^2 = 972$ $\therefore -4q^3 - 27r^2 = 1372 - 972 = 400$, which is the square of 20.

Example 2. Let $x^3 - 97x + 264 = 0$, the roots are $3, 8, -11$, $q = -97$, $r = -264$. Hence $-4q^3 - 27r^2 = 1768900$, which is the square of 1330.

Example 3. Let $x^3 - 19x + 30 = 0$, the roots are $2, 3, -5$, $q = -19$, $r = -30$. Hence $-4q^3 - 27r^2 = 3136$, which is the square of 56.

Since this function of the coefficients $\sqrt{-(4q^3 + 27r^2)}$ plays an important part in the theory of cubic equations, I propose to distinguish it by the symbol ϕ. We have therefore ϕ = product of differences of roots. As a verification, let us resume the three last examples.

Example 1. ϕ was found $= 20$. Roots were $1, 2, -3$, $\therefore$ their differences (taken in order), $-1, 5, -4$, the product of which three numbers is 20.

Example 2. ϕ was found $= 1330$. Roots were $3, 8, -11$, $\therefore$ their differences $-5, 19, -14$. And $5 \times 19 \times 14 = 1330$.

Example 3. ϕ was found $= 56$. Roots were $2, 3, -5$, $\therefore$ their differences $-1, 8, -7$. And $8 \times 7 = 56$.

These theories afford an easy solution of any cubic $x^3 + qx - r = o$, which has a root of the form $a + \sqrt{b}$; where a and b are either integers or rational fractions, and $\sqrt{b}$ is a Surd in its lowest terms.

If $a + \sqrt{b}$ is one root, the other roots will be $a - \sqrt{b}$ and $-2a$. The equation having these roots will be

$$x^3 - (3a^2 + b)\,x + 2a\,(a^2 - b) = o$$

whence

$$q = -(3a^2 + b) \text{ and } r = -2a\,(a^2 - b)\,.$$

Let us calculate the value of ϕ or $\sqrt{-(4q^3 + 27r^2)}$ in this equation. First, we have $-q^3 = (3a^2 + b)^3$ and $r^2 = 4a^2(a^4 - 2a^2b + b^2)$. Therefore we have,

$$\begin{aligned}
-\ q^3 &= 27\,a^6 + 27\,a^4b + 9\,a^2b^2 + b^3 \\
\therefore -\ 4q^3 &= 108\,a^6 + 108\,a^4b + 36\,a^2b^2 + 4b^3 \\
-\ 27r^2 &= -\,108\,a^6 + 216\,a^4b - 108\,a^2b^2 \\
\therefore -4q^3 - 27r^2 &= \quad * \quad + 324\,a^4b - 72\,a^2b^2 + 4b^3 \\
&= 4b\,(81\,a^4 - 18\,a^2b + b^2) \\
\therefore \sqrt{-(4q^3 + 27r^2)} &= \pm 2\sqrt{b}\,(9\,a^2 - b)\,.
\end{aligned}$$

But since in this equation we know the roots, a much shorter way of finding ϕ, is to take the product of the differences. Since the roots are $a + \sqrt{b}$, $a - \sqrt{b}$, $-2a$, their differences, taken in order, will be $2\sqrt{b}$, $3a - \sqrt{b}$, $-3a - \sqrt{b}$, the product of which gives $\phi = 2\sqrt{b}\,(b - 9a^2)$ the same as before. Now, by hypothesis a and b are integers or rational fractions, therefore $2(b - 9a^2)$ is rational, and therefore $\phi = \sqrt{b}$ multiplied by a rational quantity.

This conclusion is obviously of great importance, since it shows that we have only to compute the value of ϕ in the given equation $x^3 + qx - r = o$, and we obtain $\phi = \mathrm{R}\sqrt{b}$, where R is some rational quantity. The unknown quantity which we have called b separates itself naturally, so to speak, and comes to light, being left by itself under the radical sign.

And, once that b is known, a is easily found from the equation $q = -(3a^2 + b)$. And, therefore, the three roots of the equation $a + \sqrt{b}$, $a - \sqrt{b}$, and $-2a$ become known.

I will now proceed to apply this theory to some numerical examples, which will make it perfectly clear.

I will first observe that the solutions obtained resemble, in one respect at least, the theorem given in most treatises of algebra. "If an equation has two equal roots they can be found." You do not perceive from the aspect of the equation that it has two equal roots, but you are desired to make use of a certain process, and then the equal roots (if any) will appear. So here it is not apparent that an equation has roots of the form $a + \sqrt{b}$, but if it has, and $\sqrt{b}$ be in its lowest terms, both a and b can be found.

Examples of this mode of solution.

Example 1. Let $x^3 - 16x - 24 = 0$. Here $q = -16$, $r = 24$. We compute ϕ as follows:—

$$\begin{aligned} -q^3 &= 4096 \;\therefore\; -4q^3 = 16384 \\ r^2 &= 576 \;\therefore\; 27r^2 = 15552 \\ \therefore\; \phi^2 &= -4q^3 - 27r^2 = 832 \end{aligned}$$

And ϕ will be $\sqrt{832}$, which we must reduce to its lowest terms. We find that 832 is divisible 6 times by 2, the successive quotients being 416, 208, 104, 52, 26, 13. Hence $\phi^2 = 2^6 . 13$ and $\phi = 2^3\sqrt{13}$. The number 13 left under the radical sign is therefore $= b$. To find a we proceed as follows: $b = 13$, $q = -16$ $\therefore\ b + q = -3$, but this is equal to $-3a^2 \;\therefore\; a = 1$. Therefore the roots of the equation are $1 + \sqrt{13}$, $1 - \sqrt{13}$ and -2.

Example 2. Let the equation be $x^3 - 105x - 50 = 0$. Here $q = -105$, $r = 50$. Therefore $-4q^3 - 27r^2$ or ϕ^2 comes out 4563000. This is divisible by 10^3, the quotient is 4563 which is 3 times divisible by 3, the quotients being 1521, 507, 169, and the last number is seen to be the square of 13. Hence $\phi^2 = 10^3 . 3^3 . 13^2$. Extracting the square root, the rational part is $10 . 3 . 13$ or 390, and the irrational part is $\sqrt{10 . 3}$. Hence $\phi = 390\sqrt{30}$. But the rational part is of no use, and need not be calculated. It is sufficient to *omit* all numbers of which *even* powers occur in the value of ϕ^2, and instead of any *odd* power of a number to write the first power, or the number itself. Thus, instead of writing

$\phi^2 = 10^3 . 3^3 . 13^2$, it suffices to write $\phi^2 = 10 . 3$ and $\phi = \sqrt{10 . 3}$. Hence the value of b in this equation is 10×3 or 30, and since $q = -105$, $b + q = -75 = -3a^2$, $\therefore a = 5$ and the roots of the equation are $5 + \sqrt{30}$, $5 - \sqrt{30}$, -10.

Example 3. Let $x^3 - 15x + 4 = 0$, here $q = -15$, $r = -4$ and $\phi^2 = -4q^3 - 27r^2 = 13068$. This is twice divisible by 2, the quotients being 6534 and 3267, then 3 times divisible by 3, the quotients being 1089, 363, 121, and the last number is the square of 11 $\therefore \phi^2 = 2^2 . 3^3 . 11^2$, which reduces itself, by expunging the square factors, to 3. But we have seen that b is what ϕ^2 becomes after the omission of all square factors, $\therefore b = 3 \therefore b + q = 3 - 15 = -12 = -3a^2 \therefore a = 2$, and the roots are $2 \pm \sqrt{3}$, -4.

Example 4. Let $x^3 - 80x + 200 = 0$. Here $q = -80$, $r = -200$ $\therefore \phi^2 = -4q^3 - 27r^2 = 968000$. First divide by 10^2 which leaves quotient 9680, then 4 times by 2, the quotients being 4840, 2420, 1210, 605, which, divided by 5, leaves 121 or 11^2. Hence $\phi^2 = 10^2 . 2^4 . 5 . 11^2$, and omitting the square factors, we find $b = 5$. Then $b + q = 5 - 80 = -75 = -3a^2 \therefore a = 5$ and the roots are $5 \pm \sqrt{5}$, -10.

Example 5. Let $x^3 - 30x + 36 = 0$. Here $q = -30$, $r = -36$ and ϕ^2 comes out $= 73008$. Divide 4 times by 2, and the last quotient is 4563. Then divide 3 times by 3, and the last quotient 169 is seen to be the square of 13. Hence $\phi^2 = 2^4 . 3^3 . 13^2$, and omitting the square factors, $b = 3$ whence $b + q = 3 - 30 = -27 = -3a^2$, whence $a = 3$, and the roots are $3 \pm \sqrt{3}$, -6. When the coefficients are large numbers, the trouble of solving numerical equations naturally increases. But this is a mere affair of arithmetic, and the principles of solution remain unaltered. I will give an example or two with large coefficients.

Example 6. Let $x^3 - 1456\,x - 456 = 0$. Here $q = -1456$, $r = 456$ and $\phi^2 = -4q^3 - 27r^2$ comes out 12340892992, a number which is 6 times divisible by 2. Continuing the reduction, we finally obtain $\phi = 8 . 719\sqrt{373}$, whence $b = 373 \therefore b + q = 373 - 1456 = -1083 = -3a^2$ whence $a^2 = 361 \therefore a = 19$. Hence the roots are $19 \pm \sqrt{373}$, -38. Let us verify this result with regard to the root -38,

$$\begin{array}{rr} -38^3 = & -54872 \\ -456 = & -456 \\ \hline & -55328 \end{array} \qquad \begin{array}{r} 1456 \\ 38 \\ \hline 11648 \\ 4368 \\ \hline +55328 \end{array}$$

Therefore the equation is satisfied.

Example 7. Let $x^3 - 160x + 504 = 0$. Here $q = -160$, $r = -504$. Therefore $-4q^3 - 27r^2$ comes out $= 9525568$. This number is 3 times divisible by 4, and then it is found to be divisible by 13. The quotient is 11449, which a table of squares shows to be the square of 107. Hence $\phi^2 = 2^6 . 13 . 107^2$, whence omitting the square factors $b = 13$. And $b + q = 13 - 160 = -147 = -3a^2$

$\therefore a=7$, and the roots are $7 \pm \sqrt{13}, -14$. The same rules apply when the coefficients are fractional, of which I will give some examples.

Example 8. Let $$x^3 - \frac{15}{4}x - \frac{11}{4} = 0$$

Hence $$q = -\frac{15}{4} \text{ and } r = \frac{11}{4}. \text{ Hence } -4q^3 = \frac{3375}{16}$$

and $$27r^2 = \frac{3267}{16} \therefore \phi^2 = \frac{108}{16} = \frac{27}{4} \text{ and } \phi = \frac{3}{2} \cdot \sqrt{3}$$

$$\therefore b = 3 \quad \text{Hence } b + q = 3 - \frac{15}{4} = -\frac{3}{4} = -3a^2 \therefore a = \frac{1}{2},$$

and the roots are $\frac{1}{2} \pm \sqrt{3}, -1$.

Example 9. Let $$x^3 - \frac{26}{15}x + \frac{8}{135} = 0.$$

Here $-4q^3 - 27r^2$ comes out a fraction, of which the numerator is 209952 and the denominator $3 \cdot 15^3$. First let us take the latter, which may be written $3^4 \cdot 5^3$, but omitting the square factors, this reduces itself to 5. The numerator is 5 times divisible by 2, the last quotient being 6561, which $= 3^8$. Hence omitting square factors the numerator $= 2$. Therefore

$$\phi^2 \text{ or } b = \frac{2}{5}.$$

Again $$b + q = \frac{2}{5} - \frac{26}{15} = -\frac{4}{3} = -3a^2 \therefore a = \frac{2}{3}.$$

Hence the roots are $$\frac{2}{3} \pm \sqrt{\frac{2}{5}}, \ -\frac{4}{3}.$$

The same rules apply when the roots are imaginary of the form $a \pm \sqrt{-b}$, provided that $\sqrt{b}$ is a surd *in its lowest terms.*

Example 10. Let $x^3 - 22x + 84 = 0$. Here $q = -22$ $r, = -84$, whence $-4q^3 - 27r^2$ is a *negative* quantity -147920. This number is 4 times divisible by 2, and then once by 5, the last quotient is 1849, which is the square of 43. Hence $\phi^2 = 2^4 \cdot 5 \cdot 43^2$ *taken negatively*. And omitting square factors, ϕ^2 becomes $b = -5$. Hence $b + q = -5 - 22 = -27 = -3a^2 \therefore a = 3$, and the roots are $3 \pm \sqrt{-5}, -6$.

Example 11. Let $x^3 - 68x + 320 = 0$. Here $-4q^3 - 27r^2$ is negative, and $= 1507072$. This number is divisible 8 times in succession by 2, then once by 7, and the last quotient is 841, which is the square of 29, therefore $\phi^2 = 2^8 \cdot 7 \cdot 29^2$ taken negatively, and omitting square factors ϕ^2 becomes $b = -7$. Hence $b + q = -7 - 68 = -75 = -3a^2 \therefore a = 5$, and therefore the roots are $5 \pm \sqrt{-7}, -10$.

It makes no difference whether the given cubic wants the second term or possesses it. In the latter case, it must be taken away by the usual rule, which will change the root from $a + \sqrt{b}$ to $\left(a - \frac{p}{3}\right) + \sqrt{b}$ (p being the coefficient of the second term.) If p is non-divisible by 3, this will cause the new equation to have fractional coefficients. But these cause no difficulty. See examples 8 and 9.

I have sufficiently shown how roots of the form $a + \sqrt{b}$ can be found, provided that $\sqrt{b}$ is a surd in its lowest terms. I will now proceed to consider roots of the form $a + k\sqrt{b}$, where, for simplicity, I will suppose a, b, k integers, $\sqrt{b}$ a surd in its lowest terms, and that the equation wants its second term.

THEOREM.—"k is always a factor of $\frac{\phi}{2}$."

For, since ϕ is the product of the differences of the roots, if the roots are $a + k\sqrt{b}, a - k\sqrt{b}, -2a$, the differences taken in order will be $2k\sqrt{b}, 3a - k\sqrt{b}, -3a - k\sqrt{b}$, the product of which is $2k\sqrt{b}\,(k^2b - 9a^2) = \phi$, whence $\frac{\phi}{2} = k\sqrt{b}\,(k^2b - 9a^2)$, of which it is evident that k is a factor. It is also evident, that by computing the value of ϕ, $\sqrt{b}$ becomes known, although a and k continue unknown. But, since k is a factor of $\frac{\phi}{2}$, it may sometimes be easily discovered. This will best be explained by a few examples. It must be remembered, that since the roots x, y, z, are $a \pm k\sqrt{b}$, and $-2a$, the coefficient $q = xy + xz + yz = xy - (x + y)^2 = a^2 - k^2b - 4a^2 = -3a^2 - k^2b$, whence $3a^2 + k^2b = -q$.

Example 1. Let $x^3 - 245x - 482 = 0$.

Here $\phi^2 = -(4q^3 + 27r^2) = 52551752$, which, being decomposed in the usual way, we find $\frac{\phi}{2} = 11 \cdot 233\sqrt{2}$, from whence we learn that $b = 2$, and that the roots are therefore of the form $a \pm k\sqrt{2}$. Since then k is a factor of $\frac{\phi}{2}$, it must either equal 11 or 233. But it must also satisfy the general equation $3a^2 + bk^2 = -q$, which in this instance is $3a^2 + 2k^2 = 245$; and since the number 233 is evidently much too large, the true solution must be $k = 11$. Hence $3a^2 = 245 - 2k^2 = 245 - 242 = 3$, whence $a = 1$. Therefore, the three roots are $1 + 11\sqrt{2}$ $1 - 11\sqrt{2}$, and -2.

Before going further, I wish to make some general remarks. All cubic equations, whose coefficients are whole numbers, may, I believe, be divided into three classes.

(1.) All the roots integers.

(2.) Only one root an integer.

(3.) No integer roots.

It is of the second class only that I am now treating. Of course, if an equation

has *two* integer roots it must have *three*, else the coefficients would not be whole numbers. Supposing, then, that a cubic equation has one integer root, and no more, then the two other roots must be of the form $a \pm k\sqrt{b}$, where a, b, k, are integers, and $\sqrt{b}$ is a surd in its lowest terms. I have shown how the roots can be found when $k = 1$, and I am now inquiring whether they can be found when k has other values, because that would amount to a general solution of this class (which I have called the second class) of cubics. The example which I have just given shows that it can be readily effected in certain cases, but how far is the method general?

Example 2. Let $x^3 - 975x - 9972 = 0$. In this equation $\frac{\phi}{2}$ comes out $= 3 . 17 . 181 \sqrt{3}$. Whence $b = 3$ and the roots are seen to be of the form $a \pm k\sqrt{3}$. Hence, k being a factor of $\frac{\phi}{2}$, must equal either 3, 17, or 181, unless it be the product of two of them, as $3, 17 = 51$. But we have also $3\,a^2 + b\,k^2 = -q$, or $3\,a^2 + 3k^2 = 975$; whence $a^2 + k^2 = 325$. Hence it is plain that k cannot be so large a number as 181, or even as 51; therefore it must be either 3 or 17. If we try 3, we get $a^2 = 325 - 9 = 316$, which is not a square. We must therefore have $k = 17$. This gives on trial $a^2 = 325 - 289 = 36$; whence $a = 6$. The roots of the equation are therefore $6 + 17\sqrt{3}$, $6 - 17\sqrt{3}$, and -12.

It will be observed that in this instance the problem of solving the cubic is converted into that of finding two squares such that their sum $a^2 + k^2$ may $= 325$. Of course they are easily found. The general case gives $3a^2 + bk^2 = -q$, where b and q are known integers. This is easily solved by trial when the coefficient q is not too large. This, therefore, is an indeterminate problem of the second degree. If its solution is regarded as within the domain of ordinary arithmetic, the solution of cubic equations *of the second class* must be considered as effected. At any rate, the problem is transformed into a very different one. It will be observed that unless we had found the value of $\sqrt{b}$ from the properties of the function ϕ, we could not have effected the last-mentioned transformation; for b would have remained unknown. We come to the conclusion that this method reduces the solution of the *general* cubic (of the second class) to the solution of the indeterminate problem $3a^2 + bk^2 = -q$, where b and q are given.

Example 3. Resuming the former example 6 in p. 7 of this memoir $x^3 - 1456\,x - 456 = 0$, in which we found $b = 373$, the equation to be solved is $3a^2 + 373k^2 = 1456$. But here every value of k, even $k = 2$, is too large. The only remaining factor of $\frac{\phi}{2}$ is *unity*. Therefore we must have $k = 1$. The value $k = 1$ gives $3a^2 = 1456 - 373 = 1083$, which gives $a = 19$, as we found before.

Example 4. Take the former example 7, $x^3 - 160x + 504 = 0$, in which we

found $b = 13$. Hence the equation to be satisfied is $3a^2 + 13k^2 = 160$. There are two solutions, viz., $a = 7$ with $k = 1$, and $a = 6$ with $k = 2$. To decide between them, we take the root $-2a$, which is -14 in the first case, and -12 in the second case; and, substituting these numbers in the original equation, $x^3 - 160x + 504 = 0$, we find that -14 satisfies the equation, while -12 fails. Therefore the root is $7 + \sqrt{13}$. Hence we see that there may be a *plurality of solutions* of the equation $3a^2 + bk^2 = -q$; but they are easily found, when the coefficients of the given equation are of moderate magnitude.

I think it may be of some interest to add, to these examples of my method, Wood's own example, in illustration of Cardan's rule, taken from his Algebra, 6th edition, p. 172. He proposes to find the roots of $x^3 + 6x - 20 = 0$ where $q=6$, $r=20$. Since q is positive, two roots must be imaginary. Computing $-4q^3 - 27r^2 = \phi^2$ we find that it equals -11664. This number $= 2^4 . 3^6$, multiplied by -1. Hence $\phi = 2^2 . 3^3\sqrt{-1}$, and, consequently, we discover the value of $b = -1$.

The formula $3a^2 + bk^2 = -q$ becomes $3a^2 - k^2 = -6$, or $3a^2 + 6 = k^2$. It is easy to satisfy this by putting $a = \pm 1$, $k=3$. We find on trial that the negative sign is required. Therefore the roots are $-1 + 3\sqrt{-1}$, and $-1 - 3\sqrt{-1}$, and 2. To verify this, we may observe that the two former are the roots of the quadratic $x^2 + 2x + 10 = 0$, which, being multiplied by $x - 2 = 0$, gives back the proposed cubic.

Part II.—*Some properties of Cubic Equations whose Roots are whole numbers.*

Let $x^3 + qx - r = 0$ be the proposed cubic, wanting the second term. I shall suppose q to be negative, because otherwise the equation would have impossible roots. Let the roots x, y, z be whole numbers, either positive or negative.

Since $\quad x + y + z = 0, \quad$ therefore $\quad z = -(x + y)$

Hence $\quad q$, or $xy + xz + yz = xy - (x + y)^2$

or, $\quad -q = x^2 + xy + y^2.$

This may be shown in another way. Since $x^3 + qx - r = 0$, and $y^3 + qy - r = 0$. $\therefore$ by subtraction $x^3 - y^3 = -q(x - y)$, $\therefore -q = x^2 + xy + y^2$, as before.

Theorem 1.—"The coefficient q is either of the form $3n$ or $3n + 1$."

Demonstration.—All numbers are of one of the three forms, $3n$, $3n + 1$, or $3n + 2$. Therefore $x - y$ is of one of those forms. And therefore $(x - y)^2$ is of one of the two forms $3n$ or $3n + 1$. Add $3xy$, which makes no difference, since it is a multiple of 3. Therefore, $x^2 + xy + y^2$ is of the form $3n$, or else of the form $3n + 1$. Therefore the coefficient q is of one of those forms.

It is remarkable that this theorem should not be found, as I think, in the treatises of Algebra, for it furnishes a *negative test* as to whether the roots of a given equation, $x^3+qx-r=0$, are whole numbers. If, for any reason, it is supposed that they are so, let the coefficient q (taken positively) be divided by 3; then, if it leaves the remainder 2, the roots cannot be whole numbers.

Hence, if the roots of the equation *are* whole numbers, the coefficient q cannot be taken *ad libitum*. Indeed, by substituting various numbers for x and y in the formula $q=x^2+xy+y^2$, it will be seen that, of the numbers below 100, only 27 can be values of q. These numbers are 3, 7, 12, 13, 19, 21, 27, 28, 31, 37, 39, 43, 48, 49, 52, 57, 61, 63, 67, 73, 75, 76, 79, 84, 91, 93, 97. It will be seen that all these numbers are of the form $3n$ or $3n+1$, never of $3n+2$. Of all these values of q only the number 91 occurs *twice*, namely, when we suppose $x=1$, $y=9$, or else $x=5$, $y=6$. From whence we may infer, that if the roots of $x^3+qx-r=0$ are whole numbers, they may, *in most cases*, be determined from the value of q alone; but that if q has more than one value, r has the same number of values.

It is also observable that this list contains every prime number (of form $3n+1$) below 100. Does this law continue? and may we infer that integer values of x and y always exist, which will satisfy $x^2+xy+y^2=q$, q being any prime of form $3n+1$?

N.B.—Since this was written, I have tried the second *century* of numbers, or those between 100 and 199, including the latter. I find that only 28 of these can be values of q or x^2+xy+y^2, and of these only *two* occur twice, namely 133 and 147. And I find that every prime of the form $3n+1$ is found in the list; therefore the induction holds good so far. There is a well-known theorem "that every prime number of the form $4n+1$ is the sum of two squares, and in one way only." Perhaps it is true (which I only offer as a conjecture) that "every prime number of the form $3n+1$ is of the form x^2+xy+y^2, and that in one way only."

We have seen that the equation $-q=x^2+xy+y^2$, expresses the relation between any two roots x and y. The solution of this equation gives $2y+x=\sqrt{-4q-3x^2}$. But $2y+x=y-z$, because $x+y+z=0$. Whence we derive this theorem: If x be any root of the equation $x^3+qx-r=0$, the difference of the other two roots $=\sqrt{-4q-3x^2}$, which may be called R. Since, then, $y+z=-x$ and $y-z=\text{R}$, we obtain $y=\frac{\text{R}-x}{2}$, $z=\frac{-\text{R}-x}{2}$. This seems much easier than the common method, which prescribes (when one root a is known) that we should divide the equation by $x-a$, and solve the resulting quadratic. Again, since $2y+x=y-z$ is a whole number, $\sqrt{-4q-3x^2}$ must be a whole number: from which the important consequence follows, that in any equation $x^3+qx-r=0$ whose roots are integers, $-4q-3x^2$ is necessarily a square, whichever of the roots is taken for x.

Example 1. Let the roots be 3, 7, -10. Then $q = 21 - 70 - 30 = -79$, $\therefore -4q = 316$. Since the values of x are 3, 7, -10, the values of x^2 are 9, 49, 100, and those of $3x^2$ are 27, 147, 300. Hence the values of $-4q - 3x^2$ are 289, 169, 16, which are all square numbers.

Example 2. Let the roots be $x=11$, $y=13$, $z=-24$, whence $-4q=1732$. Since the values of x are 11, 13, -24, those of $-4q - 3x^2$ are—

$$\begin{aligned} 1732 - 363 &= 1369 = 37^2 \\ 1732 - 507 &= 1225 = 35^2 \\ 1732 - 1728 &= 4 \quad = 2^2. \end{aligned}$$

Extracting the square roots, and taking the three roots x, y, z, cyclically, or in regular order, we have

$$\begin{aligned} y - z &= 37 \\ z - x &= -35 \\ x - y &= -2\,. \end{aligned}$$

The sum of these three equations gives $0=0$, since $x+y+z=0$ by hypothesis.

I come now to the principal object of this paper, which is to inquire under what circumstances, or in what cases, the value of one of the roots can be found, since the two others immediately follow. We have seen that $-4q - 3x^2$ is necessarily a square, therefore, in the first place, x^2 must be such a square as *not to exceed* $\frac{-4q}{3}$; and, in the next place, whatever square N^2 is tried, it is necessary that $3N^2$, subtracted from $-4q$, should leave a square remainder. But if it were necessary to try all the squares N^2 which answer the first condition, that of not exceeding $\frac{-4q}{3}$, such a process would be impracticable from its length (except in equations with small coefficients).

While meditating upon this subject, I have met with a theorem which appears to me rather of a novel kind, and which may perhaps open some new views in the theory of equations. First, I must define what I understand by "Approximate roots." If two roots x, y, of the equation $x^3 + qx - r = 0$ which has integer roots, are so nearly equal (regard being paid to their magnitude) that $2(x+y)+1$ is greater than $\frac{(x-y)^2}{3}$, then I call them "approximate roots." Such numbers, for example, are 17 and 34, because $2(x+y)+1 = 103$, which is greater than $\frac{(x-y)^2}{3}$ or $96\frac{1}{3}$. But 17 and 35 are not "approximate numbers," because $2(x+y)+1 = 105$, which is less than $\frac{(x-y)^2}{3}$ or 108. It does not, therefore, follow that the numbers are at all nearly *equal*, because they are *approximate*. Indeed, the difference between them may be very large, provided the numbers themselves are both of them large.

This *definition* of "approximate roots" having been given, my theorem is the following:—

"If an equation $x^3 + qx - r = 0$ has integer roots, and if two of them are approximate roots, then the square root of the quantity $\frac{-4q}{3}$, *rejecting decimals*, expresses one of the roots of the equation *accurately*."

The *character* of this theorem is like those to which I have already adverted, contained in books of Algebra. "If an equation has two equal roots; or if it has two equal roots with opposite signs; or if it has three roots in arithmetical progression, these roots can be found." It is not obvious that any given equation has these peculiarities; the theorems which I have cited only say that *if it have* either of them, the roots can be found. So my theorem only says, "Approximate roots, *if they exist*, can be found." Only that it has this advantage, that, far from requiring *equal* roots, which can only seldom occur, it suffices that the roots should be *ejusdem generis* so to speak, or, in other words, not very unequal, regard being had to their magnitude.

I will first give a few numerical examples, and then investigate the theory of the subject.

Example 1. Let $x^3 - 2023x + 29478 = 0.$

Here $q = -2023 \therefore \frac{-4q}{3} = 2697\frac{1}{3}$

the square root of which is ± 51 + some decimals. It will be shown hereafter that we should choose the negative sign; therefore, rejecting the decimals, try if -51 is the root, and it will be found to succeed. The work stands as follows:—

$$2023 \times 51 = 103173 \qquad -51^3 = -132651$$

$$\begin{array}{r} 29478 \\ \hline 132651 \end{array}$$

Now to find the other two roots. We have shown that $-4q - 3x^2$ is a square $= R^2$, and that the root $x = -51$, and that $y = \frac{R-x}{2}$, $z = \frac{-R-x}{2}$.

To compute the value of R. We have $x^2 = 51^2 = 2601$.

$$\begin{array}{rl} \text{We have also } -4q = & 8092 \\ 3x^2 = & 7803 \\ \hline \text{Difference} = & 289 \end{array}$$

Hence $R = \sqrt{289} = 17$. And therefore $y = \frac{17+51}{2} = 34$, and $z = \frac{-17+51}{2} = 17 \therefore$ the three roots are -51, 34, and 17.

The success of the process is owing to 17 and 34 being *approximate* numbers, although one of them is double of the other.

Example 2. Let $x^3 - 73x + 72 = 0.$

Here $q = -73 \therefore \frac{-4q}{3} = 97\frac{1}{3}$

the square root of which is 9 + some decimals; therefore, rejecting the decimals, try -9 for the root, which succeeds. For, we find

$$\begin{array}{rl} x^3 = -729 \text{ and } -73x = & 657 \\ +72 = & 72 \\ \hline & 729 \end{array}$$

To compute the value of R, we have $x^2 = 81$, and $-4q - 3x^2 = 292 - 243 = 49$. Hence $R = \sqrt{49} = 7$, and $y = \frac{R - x}{2} = \frac{7 + 9}{2} = 8$, $z = \frac{-R - x}{2} = \frac{9 - 7}{2} = 1$. Hence the roots are 1, 8, and -9. I have given this example because the roots 1 and 8 are "approximate," though one is eight times greater than the other. This is, I believe, the extreme limit of their relative magnitudes, which does not occur in any other instance.

It will be observed also that in this example the approximate roots 1 and 8 differ by 7. This is also a kind of limit, and gives rise to the following theorem or corollary:—"Whatever the magnitude of the roots of $x^3 + qx - r = 0$, whose roots are integers, if two of them differ by 7, or by a smaller number, they can all be found by a process only requiring the extraction of the square root."

Example 3. Given $x^3 - 1477x + r = 0$, to find the roots? Here $q = -1477$. I have not deemed it necessary to give the value of r in this and the next examples, for the following reason. The values of the roots usually depend upon q alone: q being given, r has generally only one value, at any rate it has but a paucity of values, and as the solution found must always suit *one* of those values, I have omitted the consideration of it to avoid the introduction of high numbers, r being generally a much higher number than q. If an arbitrary number were suggested for r, it would only follow that such a proposed equation would not have *whole numbers* for its roots, and it is such only that we are considering at present.

Since $-q = 1477$, $\frac{-4q}{3} = 1969\frac{1}{3}$. The square root of this is 44 + decimals. Try therefore -44 for the root, and it is found to succeed.

For, since $-4q = 5908$, subtract $3 \cdot 44^2 = 5808$, and there remains 100, which is a square number. Hence $z = -44$ and $R = \sqrt{100} = 10$. Therefore $x + y = 44$, $x - y = 10$, whence $x = 27$, $y = 17$. As a verification, we find that these values give for q or $xy + xz + yz$ or $17 \cdot 27 - 17 \cdot 44 - 27 \cdot 44$ the number -1477, which was given in the proposed equation.

Example 4. Let us try higher numbers.

Suppose $x^3 - 2089168\,x + r = 0$. This value of q gives $\frac{-4\,q}{3} = 2785557\,\frac{1}{3}$.

The square root of this number is 1668 + decimals, and in fact -1668 is found to be the root. For we have $-4q = 8356672$, from which if we subtract 3 times the square of 1668 or 8346672, there remains 10000, which is a square number.

To find the other roots. Since $z = -1668$, and $R = \sqrt{10000} = 100$, $x + y = 1668$ and $x - y = 100$. Hence $x = 884$ and $y = 784$.

Investigation of the Theory of the above Method.

Since $x + y + z = 0$, one root must have an opposite sign to the other two.

To fix the ideas, let two roots x, y, be *positive*, then the third or negative root z is the largest of the three, since it equals the sum of the other two. Let the equation be as before, $x^3 + qx - r = 0$. We have found that $-q = x^2 + xy + y^2$, whence $-4q = 4x^2 + 4xy + 4y^2$. Subtract from this $3z^2 = 3(x+y)^2 = 3x^2 + 6xy + 3y^2$, and we obtain $-4q - 3z^2 = x^2 - 2xy + y^2 = (x-y)^2 = D^2$ (putting D for the *difference* of the roots). Hence we find, as before (though more directly), that if z be the greatest of the three roots $\sqrt{-4q - 3z^2} = D$, which is the difference of the two smaller and positive roots. But since $-4q$ is a positive quantity, let us put it $= +Q$. In the following lines, I think I shall be clearer, if I neglect the negative sign of z for the present, and speak only of its *magnitude*; restoring the negative sign at the end of the inquiry. We have seen that $3z^2$ always falls short of Q, by the quantity D^2. Now, let us inquire what are the conditions necessary, in order that $3(z+1)^2$ may not fall short of Q, but may exceed it? Evidently that circumstance requires that the increment $3(2z+1)$ should exceed D^2, or that $2z + 1$ should exceed $\frac{D^2}{3}$; that is, that $2(x+y) + 1 > \frac{D^2}{3}$. Accordingly, this is the definition of "approximate roots," which I have proposed in the preceding pages. Hence it follows then, that whenever the roots are approximate, although z is less than $\frac{\sqrt{Q}}{3}$ (as it always is), yet $z + 1$ is greater than $\frac{\sqrt{Q}}{3}$, that is, z is greater than $\frac{\sqrt{Q}}{3} - 1$. Let $\frac{\sqrt{Q}}{3}$ be equal to the whole number A + the decimal portion d, so that d is some positive quantity *less than unity*. Hence, we have proved that z is less than $A + d$, but greater than $A - 1 + d$; therefore, *à fortiori*, it is greater than $A - 1$. But it must be remembered that z is a whole number. How can it be greater than $A - 1$, yet less than $A + d$? Evidently in one way only; it must be *equal to* A. But A is the square root of $\frac{\sqrt{Q}}{3}$, *minus* the decimal portion d; and, therefore, restoring the value of $Q = -4q$, we have $z = \sqrt{-\frac{4q}{3}}$, *rejecting the decimals*, which is what we undertook to prove. But since z is the negative root of the equation, we must, of course, give the negative sign to this square root.

For simplicity, I have hitherto supposed that the cubic equation wants its second term, but if that is not the case, it makes no material difference, because, an equation with integer roots, having its second term, can, with a little trouble, be converted into one equally with integer roots, but wanting its second term.

The method of solution which I have pointed out is sufficiently simple. I was

desirous of knowing what degree of generality it had. With this view I made the following trial of it:—Since in the first century of numbers there are twenty-seven which can be values of q, and in the second century twenty-eight, the total is fifty-five. But since three of these are found to occur twice, I add them, which gives in all fifty-eight cases. I found by trial, that in fifty of these cases the method was successful, and that it only failed in eight. It may therefore be recommended, so long as the coefficient q is of moderate magnitude.

But what is to be done, in case the number obtained proves not to be a root? (which merely shows that there are no "approximate" roots in the equation.) Are we to abandon the solution of the equation by this method? We need not do so, for I find that the number obtained, if not the root, is generally a good approximation to the root, and that by a kind of easy *supplementary* process the true root is generally obtainable. This will be best shown by taking a numerical example.

Rule of Second Approximation.—Suppose the equation $x^3 - 1533\,x - r = 0$ to be proposed for solution. Here $-q = 1533 \therefore \frac{-4q}{3} = 2044$, the root of which is 45 + decimals. In order to try whether -45 is the root, take from $-4q = 6132$, three times 2025, or 45^2. The remainder is 57, which is not a square number, and therefore -45 is not the root. Now, to find the root, assume that -45 is an approximation to it, and proceed as follows. The differences between the successive squares 45^2, 44^2, 43^2, &c. &c., are $2 \cdot 44 + 1$, $2 \cdot 43 + 1$, &c. &c., or 89, 87, 85, 83, &c. &c. Multiply these by three, and we get the series 267, 261, 255, &c., in which each successive term diminishes by *six*.

Then taking the number 57, which was the first remainder, when -45 was tried for the root, add to it the terms of the foregoing series one by one, till a square arises, which (if the numbers are high) may give the computer the trouble of consulting "a table of squares." Thus:—

$$
\begin{array}{lll}
45 & \text{gives} & 57 \\
 & & \underline{267} \\
44 & ,, & 324 \\
 & & \underline{261} \\
43 & ,, & 585 \\
 & & \underline{255} \\
42 & ,, & 840 \\
 & & \underline{249} \\
41 & ,, & 1089 = 33^2
\end{array}
$$

Therefore, -41 is the root, and 33 is the difference of the two other roots x and y. Therefore, $x + y = 41$, and $x - y = 33$, whence $x = 37, y = 4$, therefore the roots are 4, 37, -41.

In this example the roots 4 and 37 are far from being "approximate," which is the reason why the root was not found at once.

These problems may be treated in another manner. If $x^3 + qx - r = 0$, and x, y, z, are the roots, we have seen that $-4q - 3x^2 = D^2$, where D denotes $y - z$, as before. Now, though D is unknown at present, we may make two different suppositions concerning it:—*First*, that it is *small*; *secondly*, that it is *large*, in comparison with the root x. If we make the first supposition we may try to solve the equation, put in the shape $\frac{-4q}{3} - x^2 = \frac{D^2}{3}$, by supposing (since D is small) that x^2 is that square which is nearest to $\frac{-4q}{3}$ but smaller than it. This supposition is often justified by the result, on trial. But if not, let us make the other supposition, and try to solve the equation (put in the shape $-4q - D^2 = 3x^2$), by supposing (since D is large) that D^2 is that square which is nearest to $-4q$ but smaller than it. This also very often succeeds and gives at once the value of $3x^2$, and thence of x. The first of these methods is essentially the same with that described in the preceding pages, but the second is different. If both of them are employed, I find that one or other of them solves every case of the equation $x^3 + qx - r = 0$, provided the coefficient q does not exceed 200. How much further the success of the method extends I have not yet had leisure to ascertain. I will conclude by giving an example. We find that the equation $x^3 - 1533\,x - r = 0$ was not soluble by the *first* method without the help of a "second approximation," but it is readily soluble by the *second* method, as follows:—

Since $-q = 1533$, therefore $-4q = 6132$. Assume D^2 to be the greatest square which is less than 6132, this will be 6084. Hence $-4q - D^2 = 48$. Putting this $= 3x^2$ we get $x^2 = 16$, which, being a square number, shows that we are right. Therefore $x = 4$. And since $D^2 = 6084$, $D = 78$. Hence $y + z = -4$ and $y - z = 78$, whence $y = 37$, $z = -41$, and $x = 4$.

We may therefore draw this conclusion, that "If the equation $x^3 + qx - r = 0$ has integer roots, and q is a number less than 200, the roots can always be found by the simple extraction of the square root."

XXXIX.—*On Centres, Faisceaux, and Envelopes of Homology*. By Rev. HUGH MARTIN, M.A., Member of the Mathematical Society of London, and Examiner in Mathematics in the University of Edinburgh. Communicated by Professor KELLAND.

(Read 1st April 1867.)

One of the theorems of a paper which Professor KELLAND did me the honour to read to the Society, in March 1865, opens up a field of geometrical investigation so interesting and fertile, that I venture to ask attention to some of the results of a partial examination of it in the following series of propositions. I think it right to explain, that I do not venture to expect attention to them on account of any importance attaching to them individually, but on account of their number and somewhat elegant relations. Considered individually, they may be of little importance, having no claim to rank, so to speak, among propositions of a *planetary* magnitude. But a system of *moons*, however diminutive, may become interesting if they present elegant relations among their mean motions and longitudes; and an orbit that would be grudged to a pigmy planet may be willingly accorded to a host of *planetoids*. If this is still too exalted language in which to speak of the following results, I can at least confidently affirm that they indicate a direction in which an analyst of very moderate attainments may easily discover for himself a shower of *meteors*.

It is well known that when straight lines are drawn from the angles of a triangle through any point in its plane, they intersect the sides in three points, which form the angular points of a triangle so related to the first that the intersections of their corresponding sides range in a straight line. As the triangles are said to be in homology,* we may conveniently designate the point as the Centre of Homology, and the resulting straight line as the Line of Homology,—the line represented in the former paper by $\phi(P_1)$. If we have a second point, P_2, inverse to the former, we have a second line of homology, which may be called the inverse line.

Now, if the centre of homology is subjected to motion according to a given law, or in a given curve, the line of homology displaces itself, so as, in its varying positions, to constitute a *faisceau*. The Envelope of this *faisceau* may then be inquired for. Simultaneously the inverse centre of homology will move in a

* I fear this terminology may give an aspect of pretentiousness to the paper which is far from being intended. But if I was to avoid the indefinite title, "*On a Certain Class*," &c., I confess I could find no other sufficiently descriptive.

curve that may be found; and the inverse line of homology, displacing itself, will generate another *faisceau*, whose envelope also may be sought. Farther, the lines of homology, the direct and inverse, will in general intersect, and the point of intersection, partaking of the motion of the system, will describe a locus, which may also be made the subject of inquiry. The points in which perpendiculars from the centres of homology meet their respective, direct, and inverse lines of homology will also present certain loci when the centre of homology describes any given curve, and the perpendiculars themselves will generate *faisceaux*, whose envelopes may be sought.

Farther; this system of problems may be generalised and greatly extended. The line, called the line of homology, has originated in a particular geometrical consideration; but it may be considered as unshackled from its geometrical genesis. The conception may be idealised, and conceived of as not restricted to the straight line, the curve of the first degree, but as a curve of any degree whatever. It may be a function of any degree in the variables, and of any degree in the co-ordinates of the centre as parameters; and we may thus have *faisceaux* of curves of homology of any order, and envelopes corresponding to them as before. Moreover, the general problem may be inverted, and instead of inquiring, "What envelope will the *faisceau* of homology generate when the centre of homology moves in a given curve?" we may inquire, "In what curve must the centre of homology move, in order, with a given curve of homology, to beget a given envelope?"* Or the inquiry may take yet another direction, "What must be the form of the curve of homology in order that its *faisceau*, generated by the centre of homology moving in a given curve, may produce a given envelope?" Some instances of the problem, in all these forms, will be found in the following pages. It is evident that the general inquiry may be prosecuted in such directions as would task the uttermost resources of modern discoveries in the theory of linear transformation and of canonical forms. With the exception of an instance of elimination by the aid of the Jacobian and its differential coefficients, we shall not pursue it to the necessity of laborious calculations; and shall, for the most part, restrict ourselves to the conic sections, and to such forms of these as are most manageable, and do not demand intricate elimination. These forms are,—*First*, the conic circumscribing the triangle of reference, and of the general form,

$$\frac{u}{\alpha} + \frac{v}{\beta} + \frac{w}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

Second, the conic touching the three sides of the triangle, viz.,

$$\pm (u\alpha)^{\frac{1}{2}} \pm (v\beta)^{\frac{1}{2}} \pm (w\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2);$$

* The idea of the inverse of the problem of the Envelope seems first to have occurred to Boole. See his paper—characterised by his usual high generality and beautiful originality—in "Cambridge and Dublin Mathematical Journal," vol. vii. p. 156.

Third, the conic with respect to which the triangle is self-conjugate, viz.,

$$u^2\alpha^2 + v^2\beta^2 + w^2\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (3);$$

And, *fourthly*, the conic touching two sides of the triangle of reference in the points where the third side meets them, viz.,

$$k^2\alpha^2 = \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

SECTION I.—*The Faisceau of Homology being Straight Lines.*

I. If the centre of homology move in a straight line, the *faisceau* of homology (being straight lines) will envelope a conic touching the three sides of the triangle of reference.

Let the centre of homology move in the straight line

$$l\alpha + m\beta + n\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Retaining the co-ordinates of the centre in the form in which they appeared in the former paper, f^{-1}, g^{-1}, h^{-1} (co-ordinates of P_1), and substituting these in (5), we have

$$lf^{-1} + mg^{-1} + nh^{-1} = 0 \quad . \quad . \quad . \quad . \quad . \quad (6).$$

The equation of the straight line of homology is,

$$f\alpha + g\beta + h\gamma = 0 = u \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now, by the theory of envelopes,

$$du = 0 = \alpha df + \beta dg + \gamma dh \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Differentiating (6), $\quad 0 = - lf^{-2}df - mg^{-2}dg - nh^{-2}dh \quad . \quad . \quad . \quad . \quad (9).$

Multiplying (8) by an indeterminate coefficient p, and adding (9), we have

$$(p\alpha - lf^{-2})\,df + (p\beta - mg^{-2})\,dg + (p\gamma - nh^{-2})\,dh = 0 \quad . \quad . \quad . \quad (10).$$

Now, as p is indeterminate, and as it is the ratios merely of the three parameters f, g, h, we are concerned with, reducing them virtually to two—with which the fact that they are homogeneously involved accords—we are at liberty to make two suppositions. Let these be that the coefficients of df and dg in (10) shall vanish. This makes the coefficient of dh also vanish; and we have,

$$f = \pm\sqrt{\frac{l}{p\alpha}}; \quad g = \pm\sqrt{\frac{m}{p\beta}}; \quad h = \pm\sqrt{\frac{n}{p\gamma}} \quad . \quad . \quad (11).$$

Substituting these values in (7) gives,

$$\pm (l\alpha)^{\frac{1}{2}} \pm (m\beta)^{\frac{1}{2}} \pm (n\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad (12);$$

which is the equation of a conic touching the three sides of the original triangle.

II. If the centre of homology move in a conic touching the three sides of the original triangle, the linear *faisceau* of homology will envelope the curve,

$$\pm (l\alpha)^{\frac{1}{3}} \pm (m\beta)^{\frac{1}{3}} \pm (n\gamma)^{\frac{1}{3}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (13).$$

The proof of this is, *mutatis mutandis*, the same as before. And generally,—

III. If the centre of homology move in the curve

$$\pm (l\alpha)^{\frac{1}{n}} \pm (m\beta)^{\frac{1}{n}} \pm (n\gamma)^{\frac{1}{n}} = 0 \quad . \quad . \quad . \quad . \quad (14),$$

the linear *faisceau* of homology will envelope the curve,

$$\pm (l\alpha)^{\frac{1}{n+1}} \pm (m\beta)^{\frac{1}{n+1}} \pm (n\gamma)^{\frac{1}{n+1}} = 0 \quad . \quad . \quad . \quad . \quad (15).$$

IV. If the centre of homology move in a conic with respect to which the original triangle is self-conjugate, namely,

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

the linear *faisceau* of homology will envelope the curve,

$$(l\alpha)^{\frac{2}{3}} + (m\beta)^{\frac{2}{3}} + (n\gamma)^{\frac{2}{3}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (17).$$

The proof of this is the same as before; and it follows also from III., n being taken equal to $\frac{1}{2}$.

Of course, in (16), that the conic may not be imaginary, one or other of the terms l, m, n, must be affected with the coefficient $\sqrt{(-1)}$; but the equation is maintained in this form for the sake of symmetry.

V. If the centre of homology move in a conic circumscribing the original triangle, the linear *faisceau* of homology will be a *faisceau pivotante*.

The equation of the circumscribing conic is

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Replacing α, β, γ, by the co-ordinates of the centre, we have,

$$lf + mg + nh = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

And, as before,
$$f\alpha + g\beta + h\gamma = 0 = u \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

And the result is that

$$\alpha : \beta : \gamma :: l : m : n \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

That is, the line of homology revolves round the fixed point l, m, n. This is a case of what the French writers call *courbes pivotantes*,* the curve here being of the first degree, the straight line. The number of *pivots* about which a *faisceau* of curves of the degree s revolves, or through which every member of the *faisceau pivotant* passes, is of course s^2. We shall meet with a case of this subsequently, in reference to curves of the second degree—conic sections.

VI. If the centre of homology move in the circle circumscribing the original

* The reader will find this subject elegantly treated in "*Études Analytiques sur la Théorie Général des Courbes Planes*," *par* M. FELIX LUCAS (*Paris*, 1864); where, among other results, a very pretty proposition concerning a property of conics circumscribing the same quadrilateral (due to M. LAMÉ), is thus generalised:—*Les polaires d'ordre quelconque d'un point du plan, relativement aux diverses courbes d'un faisceau pivotant, forment elles-mêmes un faisceau pivotant.*

triangle, the linear *faisceau* of homology will revolve round a point whose co-ordinates are in the ratios of the sides of the triangle.

This is an immediate corollary from the former, the equation of the circumscribing circle being, as is well known,

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

VII. If the centre of homology describe a straight line, the inverse centre will describe a conic circumscribing the original triangle; the linear *faisceau* will envelope a conic touching its three sides, while the inverse linear *faisceau* will revolve round a fixed point.

This singular system of movements is merely a combination of propositions I. and V.,—it being remembered that the co-ordinates of the two centres are mutually inverse.

VIII. If the centre of homology describe a conic touching two sides of the triangle in the points where the third side meets them, the inverse centre will describe a second conic similarly situated; the linear *faisceau* will envelope a third conic similarly situated, and the inverse linear *faisceau* a fourth conic, also similarly situated.

The equation of the conic which, by hypothesis, the centre describes is,

$$k^2 \alpha^2 = \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23);$$

or, substituting the co-ordinates of the centre,

$$k^2 gh - f^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Also

$$f\alpha + g\beta + h\gamma = 0 = u \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Differentiating as before, multiplying by an indeterminate coefficient p, &c., we have,

$$f = \frac{p\alpha}{2}\,; \quad g = -\frac{p\gamma}{k^2}\,; \quad h = -\frac{p\beta}{k^2} \quad . \quad . \quad . \quad (26).$$

And substituting in (25), we find, as the equation of the envelope of the direct linear *faisceau*,

$$k^2 \alpha^2 = 4\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Farther, while the centre moves in $k^2\alpha^2 = \beta\gamma$, it is evident, inverting the co-ordinates, that the inverse centre moves in

$$\frac{\alpha^2}{k^2} = \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

and that, by the portion of the proposition already proved, the inverse *faisceau* must envelope

$$\frac{\alpha^2}{k^2} = 4\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29);$$

which completely proves the proposition.

It is also evident that if the centre of homology now move in the direct envelope of homology (27), the direct linear *faisceau* of homology will now envelope

$$k^2\alpha^2 = 16\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

the inverse centre will move in

$$\frac{\alpha^2}{k^2} = 4\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

and the inverse linear *faisceau* will envelope

$$\frac{\alpha^2}{k^2} = 16\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (32);$$

and so on, *en suite*, while the resulting envelopes are successively taken as the curve in which the centre moves,—the n^{th} envelope of the direct *faisceau* being, of course,

$$k^2\alpha^2 = (4)^n.\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

and of the inverse *faisceau*,

$$\frac{\alpha^2}{k^2} = (4)^n.\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

IX. If the centre of homology move in the curve

$$\frac{l^2}{\alpha^2} + \frac{m^2}{\beta^2} + \frac{n^2}{\gamma^2} + \frac{mn}{\beta\gamma} + \frac{nl}{\gamma\alpha} + \frac{lm}{\alpha\beta} = 0 \quad . \quad . \quad . \quad . \quad (35),$$

the linear *faisceau* will envelope a conic section.

We have, $$l^2f^2 + m^2g^2 + n^2h^2 + mngh + nlhf + lmfg = 0 \quad . \quad . \quad (36),$$

and, $$f\alpha + g\beta + h\gamma = 0 = u \quad . \quad . \quad . \quad . \quad . \quad (37).$$

Differentiating (35) and (37), and neglecting the indeterminate coefficient, as its square will divide out in the final result, we have

$$\left.\begin{aligned} 2lf + mg + nh &= \frac{\alpha}{l}, \\ lf + 2mg + nh &= \frac{\beta}{m}, \\ lf + mg + 2nh &= \frac{\gamma}{n}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (38).$$

Eliminating g and h, and neglecting the numerical coefficient, which would divide out as merged in the indeterminate multiplier, we have,

$$\left.\begin{aligned} lf &= \frac{3\alpha}{l} - \frac{\beta}{m} - \frac{\gamma}{n}; \\ \text{or,} \quad f\alpha &= \frac{3\alpha^2}{l^2} - \frac{\alpha\beta}{lm} - \frac{\gamma\alpha}{nl}. \\ \text{Similarly,} \quad g\beta &= \frac{3\beta^2}{m^2} - \frac{\beta\gamma}{mn} - \frac{\alpha\beta}{lm}. \\ \text{And,} \quad h\gamma &= \frac{3\gamma^2}{n^2} - \frac{\gamma\alpha}{nl} - \frac{\beta\gamma}{mn}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (39).$$

Substituting these in (37), we have, as the envelope required, the conic section,

$$3\left(\frac{\alpha^2}{l^2}+\frac{\beta^2}{m^2}+\frac{\gamma^2}{n^2}\right)-2\left(\frac{\beta\gamma}{mn}+\frac{\gamma\alpha}{nl}+\frac{\alpha\beta}{lm}\right)=0 \quad . \quad (40).$$

Let it now be supposed that the centre of homology moves in this last curve, and let it be required to find the envelope of the inverse *faisceau*.

We have now, $\quad 3\{(lf)^{-2}+(mg)^{-2}+(nh)^{-2}\}-2\{(mngh)^{-1}+(nlhf)^{-1}+(lmfg)^{-1}\}=0,$

and $\quad f^{-1}.\alpha+g^{-1}.\beta+h^{-1}\gamma=0=u,$

which, with the exception of the numerical coefficients, are identical with (36) and (40), the variables to be eliminated being merely inverted, and l, m, n also inverted. Moreover, the process reverses the change of numerical coefficients, as exhibited between (35) and (40), and the result is

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2+mn\beta\gamma+nl\gamma\alpha+lm\alpha\beta=0 \quad . \quad . \quad (41);$$

a conic, and somewhat singularly related to the original curve, namely (35).

X. If the centre of homology move in a conic circumscribing the original triangle, required the locus of the intersection of the direct and inverse lines of homology.

As the point whose locus is required is on both the lines, we have simultaneously,

$$\alpha f+\beta g+\gamma h=0 \quad . \quad . \quad . \quad . \quad . \quad (42),$$

and

$$\frac{\alpha}{f}+\frac{\beta}{g}+\frac{\gamma}{h}=0 \quad . \quad . \quad . \quad . \quad . \quad (43).$$

Also the equation of the circumscribing conic is,

$$lf+mg+nh=0 \quad . \quad . \quad . \quad . \quad . \quad (44).$$

From (42) and (44), we have,

$$\frac{f}{\begin{vmatrix} m, & \beta \\ n, & \gamma \end{vmatrix}}=\frac{g}{\begin{vmatrix} n, & \gamma \\ l, & \alpha \end{vmatrix}}=\frac{h}{\begin{vmatrix} l, & \alpha \\ m, & \beta \end{vmatrix}} \quad . \quad . \quad . \quad . \quad (45),$$

and, by (43),

$$\frac{\alpha}{\begin{vmatrix} m, & \beta \\ n, & \gamma \end{vmatrix}}+\frac{\beta}{\begin{vmatrix} n, & \gamma \\ l, & \alpha \end{vmatrix}}+\frac{\gamma}{\begin{vmatrix} l, & \alpha \\ m, & \beta \end{vmatrix}}=0 \quad . \quad . \quad . \quad (46),$$

or, $\quad \dfrac{mn}{\beta\gamma}(\alpha^2-\beta^2-\gamma^2)+\dfrac{nl}{\gamma\alpha}(\beta^2-\gamma^2-\alpha^2)+\dfrac{lm}{\alpha\beta}(\gamma^2-\alpha^2-\beta^2)+l^2+m^2+n^2=0 \quad (47);$

which is the locus required—a curve of the third order. If $\alpha=0$, we have

$$(n\beta-m\gamma).(\beta^2-\gamma^2)=0 \quad . \quad . \quad . \quad . \quad (48);$$

that is to say, the curve cuts the side of the triangle in points where $\beta = \pm\gamma$, and $\beta : \gamma :: m : n$. And similarly for the intersections with the other sides. The geometrical interpretation is, that the curve cuts the sides of the triangle in the points where the bisectors of the interior and exterior opposite angles cut them; as also, in the points determined by drawing straight lines from the angles to the opposite sides respectively, through the points l, m, n; that is, the pole of the line $\{l\alpha + m\beta + n\gamma = 0\}$ in which the inverse centre of homology moves, while the direct centre describes the circumscribing conic. Considering the triangle as a curve of the third order, the $3^2 = 9$ points, in which it intersects the third-order locus thus found, are in this manner somewhat elegantly determined.

XI. If the centre of homology move in a conic touching two sides of the triangle where the third side meets them, required the locus of the intersection of the direct and inverse lines of homology.

Here we have to eliminate from the three equations,

$$\left.\begin{aligned} f^2 - k^2gh &= 0, \\ \alpha f + \beta g + \gamma h &= 0, \\ \alpha gh + \beta hf + \gamma fg &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (49);$$

and the result is,

$$\frac{(\beta^2 - \gamma^2)^2}{\alpha^2\beta\gamma} - \frac{\beta^2 + \gamma^2}{\beta\gamma} + k^2 + \frac{1}{k^2} = 0 \quad . \quad . \quad . \quad (49');$$

a curve of the fourth order, which is not altered by interchanging β and γ,—as it evidently ought not to be; nor by inverting k,—which evidently also it ought not to be, since while the centre of homology moves in

$$k^2\alpha^2 = \beta\gamma,$$

the inverse centre moves in

$$\frac{\alpha^2}{k^2} = \beta\gamma,$$

and if these are interchanged, the lines of homology are simply interchanged, and the result of elimination ought to be unaltered, as we see is the case.

XII. Required the locus of the intersection of the direct and inverse lines of homology, when the centre of homology moves in a conic with respect to which the triangle is self-conjugate.

Taking the equation of the curve in which the inverse centre moves, the square of the equation of the direct line of homology, and the equation of the inverse line cleared of fractions, we have to eliminate f, g, h, from the three following equations:—

$$u = lf^2 + mg^2 + nh^2 = 0 \quad . \quad . \quad (50);$$

$$v = \alpha^2f^2 + \beta^2g^2 + \gamma^2h^2 + 2\beta\gamma gh + 2\gamma\alpha hf + 2\alpha\beta fg = 0 \quad . \quad . \quad (51);$$

$$w = \alpha gh + \beta hf + \gamma fg = 0 \quad . \quad . \quad (52).$$

Forming the Jacobian,

$$\begin{vmatrix} \frac{du}{df}, & \frac{dv}{df}, & \frac{dw}{df} \\ \frac{du}{dg}, & \frac{dv}{dg}, & \frac{dw}{dg} \\ \frac{du}{dh}, & \frac{dv}{dh}, & \frac{dw}{dh} \end{vmatrix} = \mathrm{J} \quad . \quad . \quad . \quad (53),$$

we have,

$$(\alpha f + \beta g + \gamma h) \times \begin{vmatrix} lf, & \alpha, & \beta h + \gamma g \\ mg, & \beta, & \gamma f + \alpha h \\ nh, & \gamma, & \alpha g + \beta f \end{vmatrix} = \mathrm{J} \quad . \quad . \quad . \quad (54).$$

That is,—

$$(\alpha f + \beta g + \gamma h) \cdot \left\{ \begin{array}{l} l\alpha\beta fg + l\beta^2 f^2 - l\gamma^2 f^2 - l\gamma\alpha hf + m\beta\gamma gh + m\gamma^2 g^2 - \\ m\alpha^2 g^2 - m\alpha\beta fg + n\gamma\alpha hf + n\alpha^2 h^2 - n\beta^2 h^2 - n\beta\gamma gh \end{array} \right\} = \mathrm{J} \quad (55).$$

Multiplying the part within brackets by the coefficient αf, and differentiating with respect to f, we have

$$\left\{ \begin{array}{l} 2l\alpha^2\beta fg + 3l\alpha\beta^2 f^2 - 3l\alpha\gamma^2 f^2 - 2l\gamma\alpha^2 hf + m\alpha\beta\gamma gh + m\gamma^2\alpha g^2 - \\ m\alpha^3 g^2 - 2m\alpha^2\beta fg + 2n\gamma\alpha^2 hf + n\alpha^3 h^2 - n\alpha\beta^2 h^2 - n\alpha\beta\gamma gh \end{array} \right\} \quad (56).$$

Differentiate, with respect to f, the part within the brackets of (55) in which f appears, and multiply by $(\beta g + \gamma h)$, and we have

$$\left\{ \begin{array}{l} l\alpha\beta^2 g^2 + 2l\beta^3 fg - 2l\beta\gamma^2 fg - l\alpha\beta\gamma gh - m\alpha\beta^2 g^2 + n\alpha\beta\gamma gh + \\ l\alpha\beta\gamma gh + 2l\beta^2\gamma hf - 2l\gamma^3 hf - l\gamma^2\alpha h^2 - m\alpha\beta\gamma gh + n\gamma^2\alpha h^2 \end{array} \right\} \quad . \quad (57).$$

Hence

$$(56) + (57) = \frac{d\mathrm{J}}{df} \quad . \quad . \quad . \quad . \quad . \quad (58).$$

Now, in this sum, we find in $\frac{d\mathrm{J}}{df}$—

$$\left. \begin{array}{ll} \text{Coefficient of } & f^2 = 3l\alpha(\beta^2 - \gamma^2). \\ \quad\text{''} & g^2 = \alpha\{(l - m)\beta^2 + m(\gamma^2 - \alpha^2)\} \\ \quad\text{''} & h^2 = \alpha\{(n - l)\gamma^2 + n(\alpha^2 - \beta^2)\} \\ \quad\text{''} & gh = 0 \\ \quad\text{''} & hf = 2\gamma\{(n - l)\alpha^2 + l(\beta^2 - \gamma^2)\} \\ \quad\text{''} & fg = 2\beta\{(l - m)\alpha^2 + l(\beta^2 - \gamma^2)\} \end{array} \right\} \quad . \quad . \quad . \quad (59).$$

And symmetrically for $\frac{d\mathrm{J}}{dg}$ and $\frac{d\mathrm{J}}{dh}$.

Now, it is well known that

$$\frac{d\mathrm{J}}{df} = 0; \quad \frac{d\mathrm{J}}{dg} = 0; \quad \frac{d\mathrm{J}}{dh} = 0 \quad . \quad . \quad . \quad . \quad . \quad (60).$$

These, with $u=0$; $v=0$; $w=0$, give 6 equations in $f^2, g^2, h^2, gh, hf, fg$: whence we have the following determinant:—

$$\begin{vmatrix} l, & \alpha^2, & 0, & 3\alpha\,.\,l(\beta^2-\gamma^2), & \beta\,.\,\overline{(l-m)\alpha^2+l(\beta^2-\gamma^2)}, & \gamma\,.\,\overline{(n-l)\alpha^2+l(\beta^2-\gamma^2)} \\ m, & \beta^2, & 0, & \alpha\,.\,\overline{(l-m)\beta^2+m(\gamma^2-\alpha^2)}, & 3\beta\,.\,m(\gamma^2-\alpha^2), & \gamma\,.\,\overline{(m-n)\beta^2+m(\gamma^2-\alpha^2)} \\ n, & \gamma^2, & 0, & \alpha\,.\,\overline{(n-l)\gamma^2+n(\alpha^2-\beta^2)}, & \beta\,.\,\overline{(m-n)\gamma^2+n(\alpha^2-\beta^2)}, & 3\gamma\,.\,n(\alpha^2-\beta^2) \\ 0, & 2\beta\gamma, & \alpha, & 0, & 2\gamma\,.\,\overline{(m-n)\beta^2+m)\gamma^2-\alpha^2)}, & 2\beta\,.\,\overline{(m-n)\gamma^2+n(\alpha^2-\beta^2)} \\ 0, & 2\gamma\alpha, & \beta, & 2\gamma\,.\,\overline{(n-l)\alpha^2+l(\beta^2-\gamma^2)}, & 0, & 2\alpha\,.\,\overline{(n-l)\gamma^2+n(\alpha^2-\beta^2)} \\ 0, & 2\alpha\beta, & \gamma, & 2\beta\,.\,\overline{(l-m)\alpha^2+l(\beta^2-\gamma^2)}, & 2\alpha\,.\,\overline{(l-m)\beta^2+m(\gamma^2-\alpha^2)}, & 0, \end{vmatrix} = 0 \quad (61).$$

which is the equation of the locus required, and which I have not yet calculated out. (See *Postscript*,—where l^2, m^2, n^2, are written for l, m, n.)

Hitherto we have proceeded on the supposition of the *faisceaux* being curves of the first order,—that is, straight lines. We shall now suppose them to be conics, and we shall take the four special forms, which for brevity we shall call the circumscribing, the self-conjugate, the tri-tangent, and the bi-tangent conics.

SECTION II.—*The Faisceau of Homology being Circumscribing Conics.*

XIII. (1.) Required the envelope of a *faisceau* of circumscribing conics, while the centre of homology moves in a straight line.

Let the circumscribing conic of homology be

$$\frac{f^s}{\alpha}+\frac{g^s}{\beta}+\frac{h^s}{\gamma}=0=u \quad . \quad . \quad . \quad . \quad . \quad (62),$$

and the straight line in which the centre moves

$$l\alpha+m\beta+n\gamma=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

$$i.e. \qquad \frac{l}{f}+\frac{m}{g}+\frac{n}{h}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

Differentiating (62) and (64), introducing an indeterminate coefficient, adding, making the coefficient of the differentials to vanish as before, and substituting in (62) the values thus found for f, g, h, we have, as the envelope required,

$$\left(\frac{l^s}{\alpha}\right)^{\frac{1}{s+1}}+\left(\frac{m^s}{\beta}\right)^{\frac{1}{s+1}}+\left(\frac{n^s}{\gamma}\right)^{\frac{1}{s+1}}=0 \quad . \quad . \quad . \quad . \quad (65).$$

Giving to s the appropriate values, -2, -3, $-\frac{3}{2}$; we find as follows:—

XIV. If the centre of homology move in a straight line, the *faisceau* of circumscribing conics

$$\frac{1}{f^2\alpha}+\frac{1}{g^2\beta}+\frac{1}{h^2\gamma}=0$$

will envelope the straight line

$$l^2\alpha+m^2\beta+n^2\gamma=0 \quad . \quad . \quad . \quad . \quad . \quad (66).$$

XV. If the centre of homology move in a straight line, the *faisceau* of circumscribing conics

$$\frac{1}{f^3\alpha} + \frac{1}{g^3\beta} + \frac{1}{h^3\gamma} = 0$$

will envelope the tri-tangent conic

$$\pm (l^3\alpha)^{\frac{1}{2}} \pm (m^3\beta)^{\frac{1}{2}} \pm (n^3\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad (67).$$

XVI. If the centre of homology move in a straight line, the *faisceau* of circumscribing conics

$$\frac{1}{f^{\frac{3}{2}}\alpha} + \frac{1}{g^{\frac{3}{2}}\beta} + \frac{1}{h^{\frac{3}{2}}\gamma} = 0$$

will envelope the self-conjugate conic,

$$l^3\alpha^2 + m^3\beta^2 + n^3\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad (68).$$

XVII. (2.) Required the envelope of a *faisceau* of circumscribing conics, when the centre of homology moves in a given circumscribing conic.

In this case, instead of (64), we have

$$lf + mg + nh = 0 \quad . \quad . \quad . \quad . \quad . \quad (69).$$

If, therefore, instead of f, g, h, we take as variables their inverses, and consider s as negative, our new variables will be involved precisely as in (62), and (64) and (65) will represent the envelope we now seek, provided for s we substitute $-s$, which gives

$$(l^s\alpha)^{\frac{1}{s-1}} + (m^s\beta)^{\frac{1}{s-1}} + (n^s\gamma)^{\frac{1}{s-1}} = 0 \quad . \quad . \quad . \quad (70).$$

Therefore, giving to s the appropriate values $2, 3, \frac{3}{2}$; we find as follows :—

XVIII. If the centre of homology describe a circumscribing conic, the *faisceau* of circumscribing conics

$$\frac{f^2}{\alpha} + \frac{g^2}{\beta} + \frac{h^2}{\gamma} = 0$$

will envelope the straight line

$$l^2\alpha + m^2\beta + n^2\gamma = 0 \quad . \quad . \quad . \quad . \quad (71).$$

XIX. If the centre of homology describe a circumscribing conic, the *faisceau* of circumscribing conics

$$\frac{f^3}{\alpha} + \frac{g^3}{\beta} + \frac{h^3}{\gamma} = 0$$

will envelope the tri-tangent conic

$$\pm (l^3\alpha)^{\frac{1}{2}} \pm (m^3\beta)^{\frac{1}{2}} \pm (n^3\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad (72).$$

XX. If the centre of homology describe a circumscribing conic, the *faisceau* of circumscribing conics

$$\frac{f^{\frac{3}{2}}}{\alpha} + \frac{g^{\frac{3}{2}}}{\beta} + \frac{h^{\frac{3}{2}}}{\gamma} = 0$$

will envelope the self-conjugate conic

$$l^3\alpha^2 + m^3\beta^2 + n^3\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad (73).$$

XXI. *Cor.* From the preceding we see, that since the inverse centre of homology describes a circumscribing conic, when the direct centre describes a straight line, and *vice versa*, it follows that the direct and inverse *faisceaux* of circumscribing conics have the same envelope. Compare (66) and (71); (67) and (72); (68) and (73).

XXII. (3.) Required the envelope of a *faisceau* of circumscribing conics, when the centre of homology moves in a given self-conjugate conic.

In this case, instead of (64), we have

$$\frac{l}{f^2} + \frac{m}{g^2} + \frac{n}{h^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (74).$$

If, therefore, instead of f, g, h, we take as variables their squares, and consider s as having *half* its former value, our new variables will again be involved precisely as in (62) and (64), and (65) will represent the envelope we now seek, provided for s we read $\frac{1}{2}s$, which gives

$$\left(\frac{l^s}{\alpha^2}\right)^{\frac{1}{s+2}} + \left(\frac{m^s}{\beta^2}\right)^{\frac{1}{s+2}} + \left(\frac{n^s}{\gamma^2}\right)^{\frac{1}{s+2}} = 0 \quad . \quad . \quad . \quad (75).$$

Hence giving s the appropriate values, -4; -3; -6: we have as follows:—

XXIII. If the centre of homology move in the self-conjugate conic,

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0\,,$$

the *faisceau* of circumscribing conics

$$\frac{1}{f^4\alpha} + \frac{1}{g^4\beta} + \frac{1}{h^4\gamma} = 0$$

will envelope the straight line

$$l^2\alpha + m^2\beta + n^2\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad (76).$$

XXIV. If the centre of homology move in the same self-conjugate conic, the *faisceau* of circumscribing conics

$$\frac{1}{f^3\alpha} + \frac{1}{g^3\beta} + \frac{1}{h^3\gamma} = 0$$

will envelope another self-conjugate conic, namely,

$$l^3\alpha^2 + m^3\beta^2 + n^3\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad (77).$$

XXV. If the centre of homology move in the same self-conjugate conic, the *faisceau* of circumscribing conics

$$\frac{1}{f^6\alpha} + \frac{1}{g^6\beta} + \frac{1}{h^6\gamma} = 0$$

will envelope the tri-tangent conic

$$\pm\left(\frac{\alpha}{l^3}\right)^{\frac{1}{3}} \pm \left(\frac{\beta}{m^3}\right)^{\frac{1}{3}} \pm \left(\frac{\gamma}{n^3}\right)^{\frac{1}{3}} = 0 \quad . \quad . \quad . \quad (78).$$

XXVI. (4.) Required the envelope of a *faisceau* of circumscribing conics, when the centre of homology moves in a given bi-tangent conic, $k^2\alpha^2 = \beta\gamma$.

Here we have

$$\left.\begin{aligned} \frac{f^s}{\alpha} + \frac{g^s}{\beta} + \frac{h^s}{\gamma} &= 0 = u \\ f^2 - k^2 gh &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (79).$$

and

Hence, differentiating, and merging the factor $(s - 1)$ in the indeterminate coefficient, which we may suppress, as it divides out in the final substitution, we have

$$f^{s-2} = 2\alpha \quad . \quad . \quad . \quad . \quad . \quad (1')$$

$$g^{s-1} = -k^2 h\beta \quad . \quad . \quad . \quad . \quad . \quad (2')$$

$$h^{s-1} = -k^2 g\gamma \quad . \quad . \quad . \quad . \quad . \quad (3')$$

(2′) × (3′) $\qquad (gh)^{s-2} = k^4\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$

(2′) ÷ (3′) $\qquad \left(\frac{g}{h}\right)^s = \frac{\beta}{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5')$

(4′) gives $\qquad (gh)^s = k^{\frac{4s}{s-2}}(\beta\gamma)^{\frac{s}{s-2}} \quad . \quad . \quad . \quad . \quad (6')$

(5′) × (6′) $\qquad g^{2s} = k^{\frac{4s}{s-2}}(\beta\gamma)^{\frac{s}{s-2}} \times \frac{\beta}{\gamma} \quad . \quad . \quad . \quad (7')$

$$\frac{g^{2s}}{\beta^2} = \frac{k^{\frac{4s}{s-2}}(\beta\gamma)^{\frac{s}{s-2}}}{\beta\gamma}$$

$$= k^{\frac{4s}{s-2}}(\beta\gamma)^{\frac{2}{s-2}} \quad . \quad . \quad . \quad . \quad (8')$$

$$\frac{g^s}{\beta} = \frac{h^s}{\gamma} = \pm\, k^{\frac{2s}{s-2}}(\beta\gamma)^{\frac{1}{s-2}} \quad . \quad . \quad (9')$$

By (1′) $\qquad \frac{f^s}{\alpha} = 2^{\frac{s}{s-2}}\,\alpha^{\frac{2}{s-2}} \quad . \quad . \quad . \quad . \quad . \quad (10')$

By (79), (9′), (10′) $\qquad 2^{\frac{s}{s-2}}\,\alpha^{\frac{2}{s-2}} = \pm\, 2k^{\frac{2s}{s-2}}(\beta\gamma)^{\frac{1}{s-2}}$

$$\alpha^2 = \left(\frac{k^s}{2}\right)^2\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (80);$$

the required envelope, which is another bi-tangent conic, very neatly related to the former.

XXVII. Required the locus of the fourth point of intersection of the direct

and inverse circumscribing conics of homology, when the centre of homology describes a given circumscribing conic.

[It may be noted here, that the direct and inverse circumscribing conics of homology are two members of a *faisceau* of *courbes pivotantes*, passing through the $2^2 = 4$ points, namely, the three angles of the triangle and the point whose locus is now sought. If the direct circumscribing conic is

$$\frac{f}{\alpha} + \frac{g}{\beta} + \frac{h}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (81),$$

the inverse is

$$\frac{1}{f\alpha} + \frac{1}{g\beta} + \frac{1}{h\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (82);$$

and any member of the *faisceau pivotant* to which they belong will be represented by

$$\frac{f + \frac{\lambda}{f}}{\alpha} + \frac{g + \frac{\lambda}{g}}{\beta} + \frac{h + \frac{\lambda}{h}}{\gamma} = 0\,;$$

where λ may have any value from 0 to $\pm\infty$, the direct conic being represented when $\lambda = 0$, and the inverse conic when $\lambda = \pm\infty$].

To find the locus required, we have to eliminate f, g, h, between (81) and (82) and

$$lf + mg + nh = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (83).$$

Now, these variables, the parameters, are involved in these three equations precisely as in (42), (43), (44); and the two sets of equations are identical, if for α, β, γ we read their inverses. Hence inverting α, β, γ in (47), we get the locus required, namely,

$$mn\left(\frac{\beta\gamma}{\alpha^2} - \frac{\beta^2 + \gamma^2}{\beta\gamma}\right) + nl\left(\frac{\gamma\alpha}{\beta^2} - \frac{\gamma^2 + \alpha^2}{\gamma\alpha}\right) + lm\left(\frac{\alpha\beta}{\gamma^2} - \frac{\alpha^2 + \beta^2}{\alpha\beta}\right)$$
$$+ l^2 + m^2 + n^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (84);$$

a curve of the 6th order, and which circumscribes the triangle.

XXVIII. Required the locus of the same point when the centre of homology moves in the bi-tangent conic $k^2\alpha^2 = \beta\gamma$. Here again the elimination is the same, as in (48′) and α, β, γ have to be inverted in (49). This gives

$$\alpha^2\left(\frac{\beta^2 - \gamma^2}{\beta^3\gamma^3}\right)^2 - \frac{\beta^2 + \gamma^2}{\beta\gamma} + k^2 + \frac{1}{k^2} = 0 \quad . \quad . \quad . \quad (85),$$

and which is the same curve if the bi-tangent conic were not $k^2\alpha^2 = \beta\gamma$, but $\alpha^2 = k^2\beta\gamma$.

XXIX. Required the locus of the same point when the centre of homology moves in a self-conjugate conic.

The same consideration as that used in the last two propositions shows that this locus is to be got by inverting α, β, γ in the determinant (61).

SECTION III.—*The Faisceau of Homology being Self-Conjugate Conics.*

XXX. (1.) Required the envelope of a *faisceau* of self-conjugate conics, when the centre of homology moves in a straight line.

Here we have

$$f^s\alpha^2 + g^s\beta^2 + h^s\gamma^2 = 0 = u \quad . \quad . \quad . \quad . \quad (86),$$

and

$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0 \quad . \quad . \quad . \quad . \quad . \quad (87).$$

Proceeding as before, the envelope is

$$(l^s\alpha^2)^{\frac{1}{s+1}} + (m^s\beta^2)^{\frac{1}{s+1}} + (n^s\gamma^2)^{\frac{1}{s+1}} = 0 \quad . \quad . \quad . \quad (88).$$

Giving s appropriate values, namely, 1 ; -3 ; 3 : we have as follows :—

XXXI. If the centre of homology move in a straight line, the *faisceau* of self-conjugate conics

$$f\alpha^2 + g\beta^2 + h\gamma^2 = 0$$

will envelope, or inscribe, the quadrilateral represented by

$$\pm (l)^{\frac{1}{2}}\alpha \pm (m)^{\frac{1}{2}}\beta \pm (n)^{\frac{1}{2}}\gamma = 0 \quad . \quad . \quad . \quad . \quad (89).$$

XXXII. If the centre of homology move in a straight line, the *faisceau* of self-conjugate conics

$$\frac{\alpha^2}{f^3} + \frac{\beta^2}{g^3} + \frac{\gamma^2}{h^3} = 0$$

will envelope the circumscribing conic,

$$\frac{l^{\frac{3}{2}}}{\alpha} + \frac{m^{\frac{3}{2}}}{\beta} + \frac{n^{\frac{3}{2}}}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (90).$$

XXXIII. If the centre of homology move in a straight line, the *faisceau* of self-conjugate conics

$$f^3\alpha^2 + g^3\beta^2 + h^3\gamma^2 = 0$$

will envelope the tri-tangent conic

$$\pm (l^{\frac{3}{2}}\alpha)^{\frac{1}{2}} \pm (m^{\frac{3}{2}}\beta)^{\frac{1}{2}} \pm (n^{\frac{3}{2}}\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad (91).$$

XXXIV. (2.) Required the envelope of a *faisceau* of self-conjugate conics when the centre of homology moves in a circumscribing conic.

On the same consideration as before, s in (88) must be read $-s$, which gives the required envelope

$$\left(\frac{l^s}{\alpha^2}\right)^{\frac{1}{s-1}} + \left(\frac{m^s}{\beta^2}\right)^{\frac{1}{s-1}} + \left(\frac{n^s}{\gamma^2}\right)^{\frac{1}{s-1}} = 0 \quad . \quad . \quad . \quad (92).$$

Giving appropriate values to s, viz., -1; 3; -3: we find as follows :—

XXXV. If the centre of homology move in a conic circumscribing the triangle, the *faisceau* of self-conjugate conics

$$\frac{\alpha^2}{f} + \frac{\beta^2}{g} + \frac{\gamma^2}{h} = 0$$

will envelope, or inscribe, the quadrilateral represented by

$$\pm (l)^{\frac{1}{4}}\alpha \pm (m)^{\frac{1}{4}}\beta \pm (n)^{\frac{1}{4}}\gamma = 0 \quad . \quad . \quad . \quad . \quad (93).$$

XXXVI. If the centre of homology move in a conic circumscribing the triangle, the *faisceau* of self-conjugate conics

$$f^3\alpha^2 + g^3\beta^2 + h^3\gamma^2 = 0$$

will envelope the circumscribing conic

$$\frac{l^{\frac{2}{3}}}{\alpha} + \frac{m^{\frac{2}{3}}}{\beta} + \frac{n^{\frac{2}{3}}}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad (94).$$

XXXVII. If the centre of homology move in a conic circumscribing the triangle, the *faisceau* of self-conjugate conics

$$\frac{\alpha^2}{f^3} + \frac{\beta^2}{g^3} + \frac{g^2}{h^3} = 0$$

will envelope the tri-tangent conic

$$\pm (l^{\frac{3}{2}}\alpha)^{\frac{1}{2}} \pm (m^{\frac{3}{2}}\beta)^{\frac{1}{2}} \pm (n^{\frac{3}{2}}\gamma)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad (95).$$

XXXVIII. *Cor.* Since the inverse centre of homology moves in a circumscribing conic when the direct centre moves in a straight line, and *vice versa*, it follows, as before, that the direct and inverse *faisceaux* have the same envelope. Compare (89), (93); (90), (94); (91), (95).

XXXIX. (3.) Required the envelope of a *faisceau* of self-conjugate conics when the centre moves in a self-conjugate conic.

Here we have $\qquad f^s\alpha^2 + g^s\beta^2 + h^s\gamma^2 = 0,$

and $\qquad \frac{l}{f^2} + \frac{m}{g^2} + \frac{n}{h^2} = 0;$

and, by the consideration formerly employed, reading for s in (88) $\frac{s}{2}$ we have the envelope

$$(l^s\alpha^4)^{\frac{1}{s+2}} + (m_s\beta^4)^{\frac{1}{s+2}} + (m_s\gamma^4)^{\frac{1}{s+2}} = 0 \quad . \quad . \quad . \quad (96).$$

Giving appropriate values to s, viz., 2; -6; 6: we find as follows :—

XL. If the centre of homology move in a self-conjugate conic, the *faisceau* of self-conjugate conics

$$f^2\alpha^2 + g^2\beta^2 + h^2\gamma^2 = 0$$

will inscribe the quadrilateral

$$\pm \sqrt{l}\alpha \pm \sqrt{m}\beta \pm \sqrt{n}\gamma = 0 \quad . \quad (97).$$

XLI. If the centre of homology move in a self-conjugate conic, the *faisceau* of self-conjugate conics

$$\frac{\alpha^2}{f^6} + \frac{\beta^2}{g^6} + \frac{\gamma^2}{h^6} = 0$$

will envelope the circumscribing conic

$$\frac{l^{\frac{3}{2}}}{\alpha} + \frac{m^{\frac{3}{2}}}{\beta} + \frac{n^{\frac{3}{2}}}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (98).$$

XLII. If the centre of homology move in a self-conjugate conic, the *faisceau* of self-conjugate conics

$$f^6\alpha^2 + g^6\beta^2 + h^6\gamma^2 = 0$$

will envelope the tri-tangent conic

$$\pm\left(l^{\frac{3}{2}}\alpha\right)^{\frac{1}{2}} \pm \left(m^{\frac{3}{2}}\beta\right)^{\frac{1}{2}} \pm \left(n^{\frac{3}{2}}\gamma\right)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad (99).$$

XLIII. (4.) Required the envelope of a *faisceau* of self-conjugate conics when the centre moves in the bi-tangent conic

$$k^2\alpha^2 = \beta\gamma\,.$$

Here we have $\qquad f^s\alpha^2 + g^s\beta^2 + h^s\gamma^2 = 0 = u.$

and $\qquad f^2 - k^2gh \qquad = 0.$

Differentiating as before, suppressing the factor $(s - 1)$ as merged in the indeterminate coefficient which is suppressed because it divides out in the final substitution, we have

$$f^{s-1}\alpha^2 = 2f \;\therefore\; f^s\alpha^2 = \frac{2^{\frac{s}{s-2}}}{\alpha^{\frac{4}{s-2}}} \quad . \quad . \quad . \quad . \quad . \quad [1]$$

$$g^{s-1}\beta^2 = -\,k^2h$$

$$h^{s-1}\gamma^2 = -\,k^2g$$

$$\therefore \quad (gh)^{s-2} = \frac{k^4}{(\beta\gamma)^2} \;\therefore\; (gh)^{\frac{s}{2}} = \pm\left(\frac{k^2}{\beta\gamma}\right)^{\frac{s}{s-2}} \quad . \quad . \quad . \quad [2]$$

and $$\left(\frac{g}{h}\right)^{\frac{s}{2}} = \pm\,\frac{\gamma}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [3]$$

and $$g^s\beta^2 = h^s\gamma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [4]$$

$[2] \times [3]$ $$g^s = (k^2)^{\frac{s}{s-2}}\,.\,(\beta\gamma)^{-\frac{s}{s-2}}\,.\,\frac{\gamma}{\beta}$$

$$\therefore \quad g^s\beta^2 = (k^2)^{\frac{s}{s-2}}\,.\,(\beta\gamma)^{-\frac{2}{s-2}}$$

Substituting in u; $$\frac{2^{\frac{s}{s-2}}}{\alpha^{\frac{4}{s-2}}} = -\,2\,.\left(\frac{k^s}{\beta\gamma}\right)^{\frac{2}{s-2}}$$

$$\therefore \quad \frac{2^s}{\alpha^4} = \pm\,2^{s-2}\,.\left(\frac{k^s}{\beta\gamma}\right)^2 : \qquad \pm \text{ as } s \text{ is even or odd.}$$

If s is even $\qquad k^s\alpha^2 = 2\beta\gamma$: — the envelope required.

XLIV. It is evident that if the centre move in this last bi-tangent, the envelope will be

$$k^{\frac{s^2}{2}}\alpha^2 = 2^{\frac{s+2}{2}}\beta\gamma.$$

If, again, in this, the envelope will be

$$k^{\frac{s^3}{4}}\alpha^2 = 2^{\frac{s^2+2s+4}{4}}\beta\gamma.$$

And so on *en suite;* the law being obvious, namely, that the t^{th} envelope will be

$$k^{\frac{s^t}{2^{t-1}}}\alpha^2 = \left(2.^{\frac{s^{t-1}+2s^{t-2}+4s^{t-3}+8s^{t-4}+\&c.}{2^{t-1}}}\right)\beta\gamma \quad . \quad . \quad . \quad (100).$$

or

$$\frac{k^{2.\left(\frac{s}{2}\right)^t}}{2.^{\frac{1-\left(\frac{s}{2}\right)^t}{1-\frac{s}{2}}}} . \alpha^2 = \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100').$$

—a somewhat curious result.

SECTION IV.—*The Faisceau of Homology being Tri-tangent Conics.*

XLV. (1.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a straight line.

Here we have

$$\left.\begin{aligned} \pm f^s . \alpha^{\frac{1}{2}} \pm g^s\beta_{\frac{1}{2}} \pm h^s\gamma^{\frac{1}{2}} &= 0 \\ \frac{l}{f} + \frac{m}{g} + \frac{n}{h} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (101).$$

and the envelope, found as before, is

$$\pm (l^{2s}.\alpha)^{\frac{1}{2(s+1)}} \pm (m^{2s}.\beta)^{\frac{1}{2(s+1)}} \pm (n^{2s}\gamma)^{\frac{1}{2(s+1)}} = 0 \quad . \quad . \quad . \quad (101').$$

Hence, giving to s the appropriate values, namely, $-\frac{1}{2}$; $-\frac{3}{2}$; $-\frac{3}{4}$: we find as follows:—

XLVI. If the centre of homology move in a straight line, the *faisceau* of tri-tangent conics

$$\pm\sqrt{\frac{\alpha}{f}} \pm \sqrt{\frac{\beta}{g}} \pm \sqrt{\frac{\gamma}{h}} = 0$$

will envelope the straight line

$$\pm \frac{\alpha}{l} \pm \frac{\beta}{m} \pm \frac{\gamma}{n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (102).$$

XLVII. If the centre of homology move in a straight line, the *faisceau* of tri-tangent conics

$$\pm\sqrt{\frac{\alpha}{f^3}} \pm \sqrt{\frac{\beta}{g^3}} \pm \sqrt{\frac{\gamma}{h^3}} = 0$$

will envelope the circumscribing conic

$$\frac{l^3}{\alpha} + \frac{m^3}{\beta} + \frac{n^3}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (103).$$

XLVIII. If the centre of homology move in a straight line, the *faisceau* of tri-tangent conics

$$\pm \frac{\alpha^{\frac{1}{4}}}{f^{\frac{1}{4}}} \pm \frac{\beta^{\frac{1}{4}}}{\gamma^{\frac{1}{4}}} \pm \frac{\gamma^{\frac{1}{4}}}{h^{\frac{1}{4}}} = 0$$

will envelope the self-conjugate conic

$$\frac{\alpha^2}{l^3} + \frac{\beta^2}{m^3} + \frac{\gamma^2}{n^3} = 0 \quad . \quad . \quad . \quad . \quad . \quad (104).$$

XLIX. In general; in order that the envelope may be of the form

$$\lambda\alpha^t + \mu\beta^t + \nu\gamma^t = 0$$

we must evidently have $\dfrac{1}{2s+2} = t$ or $s = \dfrac{1-2t}{2t}$

And the *faisceau*

$$\pm f^{\frac{1-2t}{2t}} \alpha^t \pm g^{\frac{1-2t}{2t}} \beta^t \pm h^{\frac{1-2t}{2t}} \gamma^t = 0$$

gives the envelope

$$\pm l^{1-2t} . \alpha^t \pm m^{1-2t} . \beta^t \pm n^{1-2t} \gamma = 0 \quad . \quad . \quad . \quad (105).$$

L. (2.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a conic circumscribing the triangle.

Here, by the same consideration as we have already employed in like cases, the envelope required is got from (101′) by reading $-s$ for s, namely,

$$\pm \left(\frac{l^{2s}}{\alpha}\right)^{\frac{1}{2(s-1)}} \pm \left(\frac{m^{2s}}{\beta}\right)^{\frac{1}{2(s-1)}} \pm \left(\frac{n^{2s}}{\gamma}\right)^{\frac{1}{2(s-1)}} = 0 \quad . \quad . \quad (106).$$

Hence, giving s the appropriate values $\frac{1}{2}$; $\frac{3}{2}$; $\frac{3}{4}$: we find as follows:—

LI. If the centre of homology move in a circumscribing conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{f\alpha} \pm \sqrt{g\beta} \pm \sqrt{h\gamma} = 0$$

will envelope the straight line

$$\pm \frac{\alpha}{l} \pm \frac{\beta}{m} \pm \frac{\gamma}{n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (107).$$

LII. If the centre of homology move in a circumscribing conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{f^3\alpha} \pm \sqrt{g^3\beta} \pm \sqrt{h^3\gamma} = 0$$

will envelope the circumscribing conic

$$\frac{l^3}{\alpha} + \frac{m^3}{\beta} + \frac{n^3}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (108).$$

LIII. If the centre of homology move in a circumscribing conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{f^{\frac{3}{2}}\alpha} \pm \sqrt{g^{\frac{3}{2}}\beta} \pm \sqrt{h^{\frac{3}{2}}\gamma} = 0$$

will envelope the self-conjugate conic

$$\frac{\alpha^2}{l^3} + \frac{\beta^2}{m^3} + \frac{\gamma^2}{n^3} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (109).$$

LIV. (3.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves also in a given tri-tangent conic, say,

$$\pm\, l\alpha^{\frac{1}{2}} \pm m\beta^{\frac{1}{2}} \pm n\gamma^{\frac{1}{2}} = 0.$$

Here we have

$$\left.\begin{array}{l} \pm f^s\alpha^{\frac{1}{2}} \pm g^s\beta^{\frac{1}{2}} \pm h^s\gamma^{\frac{1}{2}} = 0 \\ \pm \dfrac{l}{f^{\frac{1}{2}}} \pm \dfrac{m}{g^{\frac{1}{2}}} \pm \dfrac{n}{h^{\frac{1}{2}}} = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \qquad (110).$$

and

Now, $f^{\frac{1}{2}}$, $g^{\frac{1}{2}}$, $h^{\frac{1}{2}}$ are involved in equations (110) precisely as f, g, h are involved in equations (101), provided in equations 101 we substitute $2s$ for s. Making, therefore, this substitution in (101′), we have the envelope now required, namely,

$$\pm\, (l^{4s}\alpha)^{\frac{1}{2(2s+1)}} \pm (m^{4s}\beta)^{\frac{1}{2(2s+1)}} \pm (n^{4s}\gamma)^{\frac{1}{2(2s+1)}} = 0 \quad . \quad . \quad . \qquad (111).$$

Hence, giving s the appropriate values, $-\frac{1}{4}$; $-\frac{3}{4}$; $-\frac{3}{8}$: we find as follows:—

LV. If the centre of homology move in a tri-tangent conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f^{\frac{1}{2}}}} \pm \sqrt{\frac{\beta}{g^{\frac{1}{2}}}} \pm \sqrt{\frac{\gamma}{h^{\frac{1}{2}}}} = 0$$

will envelope the straight line

$$\pm \frac{\alpha}{l} \pm \frac{\beta}{m} \pm \frac{\gamma}{n} = 0 \quad . \quad . \quad . \quad . \quad . \qquad (112).$$

LVI. If the centre of homology move in a tri-tangent conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f^{\frac{3}{2}}}} \pm \sqrt{\frac{\beta}{g^{\frac{3}{2}}}} \pm \sqrt{\frac{\gamma}{h^{\frac{3}{2}}}} = 0$$

will envelope the circumscribing conic

$$\pm \frac{l^3}{\alpha} \pm \frac{m^3}{\beta} \pm \frac{n^3}{\gamma} = 0 : \quad . \quad . \quad . \quad . \qquad (113).$$

LVII. If the centre of homology move in a tri-tangent conic, the *faisceau* of tri-tangent conics

$$\pm \sqrt{\frac{\alpha}{f^{\frac{3}{4}}}} \pm \sqrt{\frac{\beta}{g^{\frac{3}{4}}}} \pm \sqrt{\frac{\gamma}{h^{\frac{3}{4}}}} = 0$$

will envelope the self-conjugate conic

$$\pm\frac{\alpha^2}{l^3}\pm\frac{\beta^2}{m^3}\pm\frac{\gamma^2}{n^3}=0 \quad . \quad . \quad . \quad . \quad . \quad (114).$$

LVIII. (4.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a bi-tangent conic, say

$$k^2\alpha^2 = \beta\gamma.$$

Here we have

$$\pm f^s\alpha^{\frac{1}{3}}\pm g^s\beta^{\frac{1}{3}}\pm h^s\gamma^{\frac{1}{3}}=0,$$

and

$$f^2 - k^2 gh \qquad = 0.$$

Proceeding as before, the envelope is found to be

$$\left(\frac{k^s}{2}\right)^4\alpha^2=\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (115).$$

another bi-tangent elegantly related to the former.

LIX. If the centre of homology move in this bi-tangent, the bi-tangent envelope will of course be

$$\left(\frac{k^{2s^2}}{2^{2s+1}}\right)^4\alpha^2=\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (116).$$

If in this, the envelope will be

$$\left(\frac{k^{4s^3}}{2^{4s^2+2s+1}}\right)^4\alpha^2=\beta\gamma \quad . \quad . \quad . \quad . \quad (117).$$

If in this, the envelope will be

$$\left(\frac{k^{8s^4}}{2^{8s^3+4s^2+2s+1}}\right)^4\alpha^2=\beta\gamma \quad . \quad . \quad . \quad . \quad (118).$$

And so on, *en suite*, the law being obvious,—the t^{th} envelope being

$$\left(\frac{k^{\frac{(2s)^t}{2}}}{2^{\frac{1-(2s)^t}{1-2s}}}\right)^4.\,\alpha^2=\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (118').$$

LX. (5.) Required the envelope of a *faisceau* of tri-tangent conics when the centre of homology moves in a self-conjugate conic, say $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$.

By a similar consideration to that used in LIV., in (101′) for s read $\frac{1}{2}s$, and we have the envelope required, namely

$$\pm(l^s\alpha)^{\frac{1}{s+2}}\pm(m^s\beta)^{\frac{1}{s+2}}\pm(n^s\gamma)^{\frac{1}{s+2}}=0 \quad . \quad . \quad . \quad . \quad (119).$$

Hence, giving to s appropriate values, namely, -1; $-\frac{3}{2}$; -3: we find as follows:—

LXI. If the centre of homology move in a self-conjugate conic, the *faisceau* of tri-tangent conics

$$\pm\frac{\alpha^{\frac{1}{4}}}{f} \pm \frac{\beta^{\frac{1}{4}}}{g} \pm \frac{\gamma^{\frac{1}{4}}}{h} = 0$$

will envelope the straight line

$$\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (120).$$

LXII. If the centre of homology move in a self-conjugate conic, the *faisceau* of tri-tangent conics

$$\pm\frac{\alpha^{\frac{1}{3}}}{f^3} \pm \frac{\beta^{\frac{1}{3}}}{g^3} \pm \frac{\gamma}{h^3} = 0$$

will envelope the circumscribing conic

$$\frac{l^3}{\alpha} + \frac{m^3}{\beta} + \frac{n^3}{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (121).$$

LXIII. If the centre of homology move in a self-conjugate conic, the *faisceau* of tri-tangent conics

$$\pm\sqrt{\frac{\alpha}{f^3}} \pm \sqrt{\frac{\beta}{g^3}} \pm \sqrt{\frac{\gamma}{h^3}} = 0$$

will envelope the self-conjugate conic

$$\frac{\alpha^2}{l^3} + \frac{\beta^2}{m^3} + \frac{\gamma^2}{n^3} = 0 \quad . \quad (122).$$

SECTION V.—*The Faisceau of Homology being Bi-tangent Conics.*

LXIV. (1.) Required the envelope of a *faisceau* of bi-tangent conics when the centre of homology moves in a straight line.

Let the *faisceau* be

$$\left.\begin{aligned} f^{2s}\alpha^2 - (gh)^s\beta\gamma &= 0 \\ l\alpha + m\beta + n\gamma &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (123).$$

and the straight line

The envelope is

$$\alpha^2 = \left(\frac{4mn}{l^2}\right)^s\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (124).$$

LXV. (2.) Required the envelope of a *faisceau* of bi-tangent conics when the centre of homology moves in a circumscribing conic. The envelope is

$$\alpha^2 = \left(\frac{l^2}{4mn}\right)^s\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (125).$$

LXVI. (3.) Required the envelope of a *faisceau* of bi-tangent conics when the centre of homology moves in a self-conjugate conic. The envelope is

$$\alpha^2 = \left(\frac{l^2}{4mn}\right)^{\frac{s}{s+2}}\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (126).$$

LXVII. (4.) Required the envelope of a *faisceau* of bi-tangent conics when the centre of homology moves in a tri-tangent conic. The envelope is

$$\alpha^2 = \left(\frac{4mn}{l^2}\right)^{\frac{2s}{2s+1}} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (127).$$

LXVIII. (5.) Required the envelope of a *faisceau* of bi-tangent conics when the centre of homology moves in the bi-tangent conic

$$\alpha^2 = k^2 \beta\gamma.$$

The envelope is

$$\alpha^2 = k^{2^s} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (128).$$

If the centre of homology move in this conic, the envelope is

$$\alpha^2 = k^{2^{s^2}} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (129).$$

If in this, the envelope is

$$\alpha^2 = k^{2^{s^3}} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (130).$$

And the t^{th} envelope is

$$\alpha^2 = k^{2^{s^t}} \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (131).$$

If we equate the indices (inverted when necessary) of (124) and (126); (125) and (126); (124) and (127); (125) and (127); (126) and (127); we get the following somewhat elegant propositions:—

LXIX. Whether the centre moves in a straight line or in a self-conjugate conic, the *faisceau* of bi-tangent conics

$$(gh)^3 \alpha^2 = f^6 \beta\gamma$$

envelopes the same bi-tangent conic

$$(4mn)^3 \alpha^2 = l^6 \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (132).$$

LXX. Whether the centre moves in a circumscribing or in a self-conjugate conic, the *faisceau* of bi-tangent conics

$$gh\alpha^2 = f^2 \beta\gamma$$

envelopes the same bi-tangent conic

$$l^2 \alpha^2 = 4mn \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (133).$$

LXXI. Whether the centre moves in a straight line or in a tri-tangent conic, the *faisceau* of bi-tangent conics

$$f\alpha^2 = \sqrt{gh} \,.\, \beta\gamma$$

envelopes the same bi-tangent conic

$$l\alpha^2 = 2\sqrt{mn} \,.\, \beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (134).$$

LXXII. Whether the centre moves in a circumscribing conic or in a tri-tangent conic, the *faisceau* of bi-tangent conics

$$(gh)^{\frac{3}{2}}\alpha^2 = f^3\beta\gamma$$

envelopes the same bi-tangent conic

$$l^3\alpha^2 = (4mn)^{\frac{3}{2}}\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (135).$$

LXXIII. Whether the centre of homology moves in a self-conjugate or a tri-tangent conic, the *faisceau* of bi-tangent conics

$$(gh)^{\frac{1}{3}}\alpha^2 = f^{\frac{2}{3}}\beta\gamma$$

envelopes the same bi-tangent conic

$$(l^2)^{\frac{1}{3}}\alpha^2 = (4mn)^{\frac{1}{3}}\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (136).$$

I have only to add, that while the general problem of this paper becomes very difficult in the case of the general equation of the second degree, by reason of the complicated elimination that is necessary, I have no doubt that this difficulty might be evaded by reducing the general equation to one or other of the forms of the conic which have engaged our attention, by an appropriate transformation of the triangle of reference. If leisure permit, I should be glad to verify this supposition, as well as give certain extensions of the theory, in another paper.

Postscript.—While this is passing through the press, I find that SILVESTER'S "Dialytic method of Elimination as applied to a Ternary System of Equations" (see "Cambridge Mathematical Journal," vol. ii. p. 234), may be used more efficiently than the Jacobian in the last problem of Section I. Writing the equations (50), (52), (51), thus:—

$$0 = l^2.f^2 + m^2.g^2 + n^2.h^2 + \; * \quad * \quad * \quad . \quad . \quad [1],$$

$$0 = \; * \quad * \quad * \; + \alpha.gh + \beta.hf + \gamma.fg \quad . \quad . \quad [2],$$

$$0 = \alpha.f + \beta.g + \gamma.h \quad . \quad . \quad [3];$$

we have to find four equations in f^2, g^2, h^2, gh, hf, fg. Re-write the three just given (taking twice the second) thus:—

$$0 = (l^2f).f + (m^2g).g + (n^2h).h \quad . \quad . \quad [4],$$

$$0 = (\beta h+\gamma g).f + (\gamma f+\alpha h).g + (\alpha g+\beta f).h \quad . \quad . \quad [5],$$

$$0 = (\alpha).f + (\beta).g + (\gamma).h \quad . \quad . \quad [6].$$

Eliminating f, g, h, we have, dropping the parentheses,

$$\begin{vmatrix} l^2f & m^2g & n^2h \\ \beta h+\gamma g & \gamma f+\alpha h & \alpha g+\beta f \\ \alpha & \beta & \gamma \end{vmatrix} = 0 \quad . \quad . \quad . \quad [7];$$

that is:—

$$0 = l^2(\beta^2-\gamma^2)\,.\,f^2 + m^2(\gamma^2-\alpha^2)\,.\,g^2 + n^2(\alpha^2-\beta^2)\,.\,h^2 + (m^2-n^2)\beta\gamma\,.\,gh + (n^2-l^2)\gamma\alpha\,.\,hf + (l^2-m^2)\alpha\beta\,.\,fg \quad [8].$$

Multiplying [3] successively by f, g, h, we have

$$\begin{array}{llllllll} 0 = \alpha\,.\,f^2 & * & * & * & +\ \gamma\,.\,hf & +\ \beta\,.\,fg & . \quad . & [9]; \\ 0 = \quad * & \beta\,.\,g^2 & + & \gamma\,.\,gh & + & \alpha\,.\,fg & . \quad . & [10]; \\ 0 = \quad * & * & \gamma\,.\,h^2 & +\ \beta\,.\,gh & +\ \alpha\,.\,hf & * & . \quad . & [11]. \end{array}$$

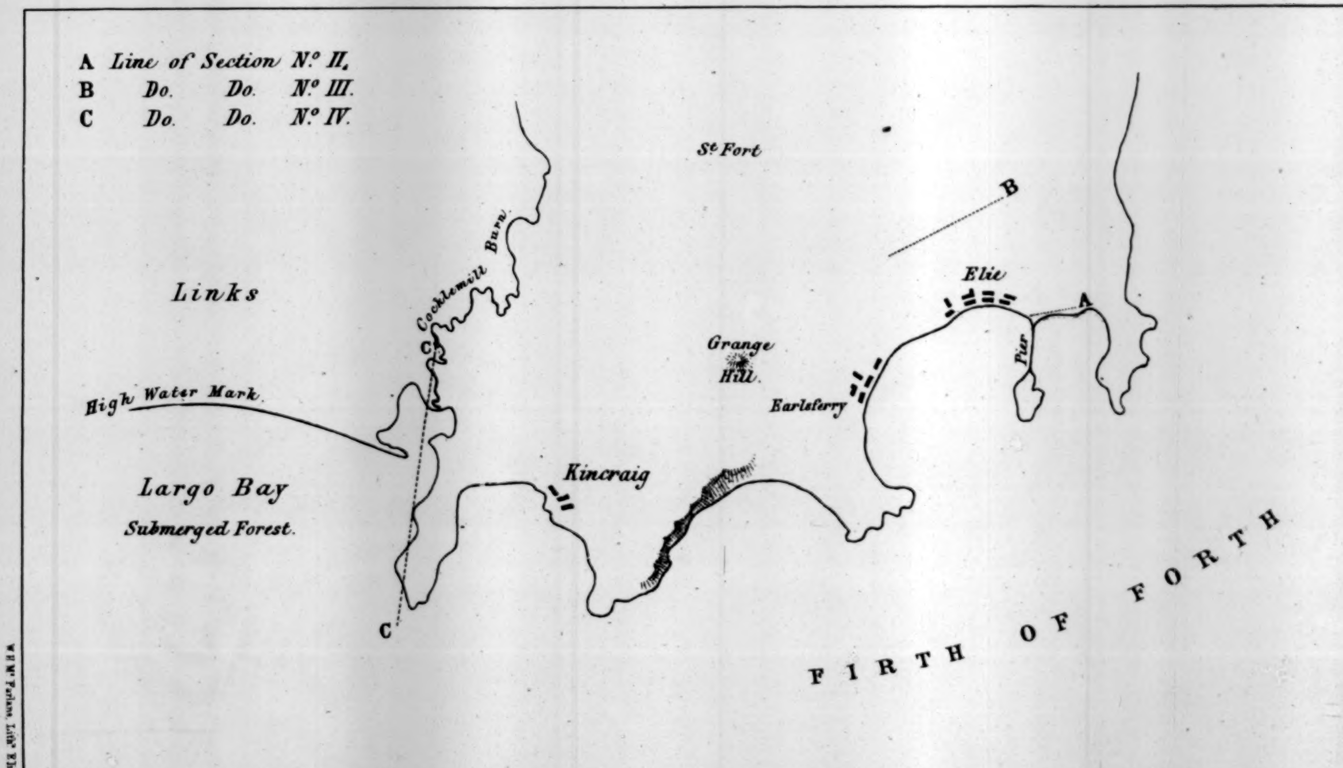
Forming now the determinant of elimination from equations [1], [2], [8], [9], [10], [11], we have, cleared of extraneous factors, the equation of the locus sought, namely,

$$\begin{vmatrix} l^2 & 0 & l^2(\beta^2-\gamma^2) & \alpha & 0 & 0 \\ m^2 & 0 & m^2(\gamma^2-\alpha^2) & 0 & \beta & 0 \\ n^2 & 0 & n^2(\alpha^2-\beta^2) & 0 & 0 & \gamma \\ 0 & \alpha & (m^2-n^2)\beta\gamma & 0 & \gamma & \beta \\ 0 & \beta & (n^2-l^2)\gamma\alpha & \gamma & 0 & \alpha \\ 0 & \gamma & (l^2-m^2)\alpha\beta & \beta & \alpha & 0 \end{vmatrix} = 0,$$

which is, when expanded, the very symmetrical curve,

$$\left(\frac{\beta^2+\gamma^2-\alpha^2}{l\beta\gamma}\right)^2 + \left(\frac{\gamma^2+\alpha^2-\beta^2}{m\gamma\alpha}\right)^2 + \left(\frac{\alpha^2+\beta^2-\gamma^2}{n\alpha\beta}\right)^2 = 4\left(\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}\right) - \left(\frac{l^2+m^2+n^2}{lmn}\right)^2.$$

It seems scarcely necessary to add that in equation [1] or (50), the curve in which the *inverse* centre moves, is *legitimately* taken, rather than that in which the centre itself moves, since in the latter case the other two equations change places, and the same elimination has to be effected, the variables being merely inverted.



W. H. Mᶜ Farlane, Lithʳ Edinʳ

Reduced tracing of the Elie Coast-line from the Ordnance Survey Map
Six inch Scale.

XL.—*On the Arctic Shell-Clay of Elie and Errol, viewed in connection with our other Glacial and more recent Deposits.* By the Rev. THOMAS BROWN, F.R.S.E. (Plate XXXVII).

(Read 4th March 1867.)

My attention was called to the subject of this paper in May 1862, during a short stay at Elie on the coast of Fife. Close to the friend's house with whom I lived a part of the sea-bank had been laid open by the waves, and among other deposits I found a bed of clay containing fossil shells, such as now live only in the Polar seas. An account of this I had the honour of laying before the Society on the 2d of March 1863. During the following autumn, while residing at Bridge of Earn, I found a similar deposit, with the same species of shells, at Errol on the Tay, and a notice of this I also laid before the Society on the 2d of May 1864, intimating that at some future period I should again ask their attention to the facts thus ascertained and the inferences to be drawn from them. The delay that has taken place has arisen from other occupations, which leave little time for such pursuits, but it has not been wholly without advantage. The cuttings of the East of Fife Railway were carried past the outskirts of Elie, and I had an opportunity of examining the series of beds, while laid open for the time, in a very remarkable way. This last autumn also, while residing on the spot for a few days, I examined with some care a transverse section, nearly at right angles to the two former; and now, in this paper, I shall endeavour first to state in detail the facts connected with these separate localities, and then to bring into one view the general results.

The Errol Section.

The great feature of the Errol district is the level clay of the Carse of Gowrie, so valuable to the agriculturist. Inland, ridges of boulder-clay are found rising from below the carse lands, and on the slope of one of these, near its base, lies this deposit with its Arctic shells. The aspect of it is very different from that at Elie, where the dark colour of the carboniferous shales shows itself in the darkness of the clay, while at Errol the colour is light or reddish, from the red

sandstones of the district. The shell-bearing clay is well laid open, occupying the whole side of the brickfield, where it shows the following series:—

Section I.

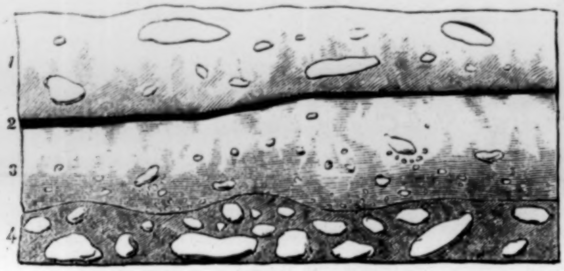


Fig. 1.

1. Immediately beneath the surface a yellowish or brown clay, from 5 to 8 feet thick, lighter in colour, and less compact in structure than the underlying beds.

2. A black band of particularly fine clay, a foot or rather less in thickness.

3. An underlying deposit of dark-red tenacious clay, from 5 to 8 feet in depth.

4. The boulder-clay or till, on which all the above beds rest. Its uneven surface shows that it had been denuded, and there is a marked line of separation between it and the next bed above.

The whole of this series contains boulders and stones, from the size of a marble up to masses of very considerable weight, some of the largest being found near the top of the section. The clay itself is particularly fine, but it requires much care to free it from stones,—every separate portion, as it goes through the machine, requiring to be hand-picked. The fossil shells are found all through the three highest beds. In No. 2 they are particularly abundant, the blackness of the bed being due to the decay of animal matter. They are also very plentiful near the bottom of No. 3, where they are found clustering around and beneath the enclosed boulders—a fact which seems to show that at the time these shells lived this part of the sea-bottom must have been swept by a strong current, and they had found it convenient to shelter themselves under the lee of the stones.

The shells themselves are in beautiful preservation. Though sometimes fractured by pressure or shrinkage, yet the epidermis and other parts are usually as complete as if they were recent. They soon crumble on exposure to the air. I shall afterwards refer to the species, and to the skeleton of a seal found along with them.

The position of the three sections at Elie will be understood by referring to the accompanying sketch map. (Plate XXXVII.)

Elie Shore Section.

This occurs close to the harbour, running eastwards from the pier along high-water mark, where, in 1862, the beds were much more fully laid open than they are now. In the descending order they showed the following series:—

Section II.

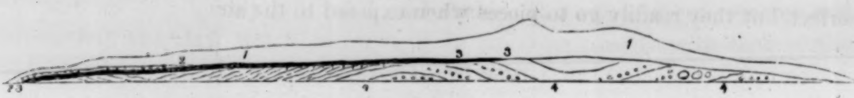


Fig. 2.

1. Immediately below the surface blown sand, in horizontal layers, from 4 to 6 feet, or rather more, in depth. This is the seaward edge of a great sheet of the same kind, which stretches, with intervals, far along the coast, and reaches in some places more than a mile inland. The whole town of Elie is built on it. In the present section, at the point where beds 2 and 3 thin out, the lowest portion of this blown sand has hardened into a kind of sandstone or concrete, rolled fragments of which may be found among the boulders of the beach

2. A thin bed of sea shingle and shells. Apparently it is a portion of the so-called raised beach, and will be referred to more at length under Section IV.

3. A layer of peat from 5 to 10 inches in thickness. It is *in situ*, the plants which formed it having grown on the spot, as is shown by the traces of their roots going down vertically into the underlying beds. The roots themselves are decayed, but the ferruginous stains, which show their course, are very marked. Of the remains enclosed in this peat, by far the most common is the *Arundo phragmites*, the common reed of our fresh-water marshes, which grows at present freely in the neighbourhood. The thin outer coverings of the underground stems, with their unmistakable jointings, occur in great quantities. There are seeds of various kinds, and decayed wood, but no shells. This seems a portion of the submerged forest of this coast.

At this point there occurs a break in the series which should be carefully noted. These three upper deposits are markedly unconformable to those beneath. When first laid down the lower beds would be approximately horizontal; but, by the action of some force, they must have been thrown into convolutions. The process of deposition was suspended; part of the material already laid down was abraded and swept away; and when, after a time, the process of deposition was again resumed, the newer beds were laid down horizontally across the truncated and upturned edges of the lower beds. This break should be noted as showing a blank in the record.

4. The deposit in which the fossil shells occur. The upper portion of it consists of layers of sand, with partings of finely triturated coal shale, the sand being of an ochreous or deep brown colour. In this portion I observed no fossils. Gradually, on passing downwards, the layers begin to get argillaceous till towards the base, and in the eastern half of the section, the deposit passes into a peculiarly stiff, tenacious, unstratified clay. It is here the shells occur, and in considerable quantities. All the portions of the shells are in many cases beautifully perfect, but they readily go to pieces when exposed to the air.

Elie Inland Section.

This was laid open in the railway cutting, beginning at a point to the east of the railway station, and going west to the bridge beside the schoolhouse. It is distant about the third of a mile from the preceding section, and showed, when first fully laid open, the following series of deposits from above downwards:—

Section III.

Fig. 3.

1. Blown sand of very considerable depth. All through it contains numerous darker lines, showing former surfaces, and containing land shells, especially the *Succinea putris.* Besides these there were intercalated, at four or five different levels, distinct beds of peat, *a b c d e,* the uppermost of which is 6 feet in depth. They are full of land and fresh-water shells, to which I shall afterwards refer.

2. A bed of peat, from 6 to 10 inches thick. The structure is markedly different from that of the overlying peat beds, more earthy, and bearing a close resemblance to No. 3 of the previous section. Like that it contained no shells. The only organisms I detected in it were the roots of *Equisetum,* which, like those of the *Arundo* in the peat of the shore, indicate a fresh-water origin.

3. A deposit of gravel and sand, with portions of clay all arranged in layers, which are nearly horizontal, but with a gentle dip eastwards, accommodating themselves to the general lie of the ground. To the west of the schoolhouse, where the ground slopes the other way, the inclination of these layers is also reversed, showing that, when this deposit was laid down, the contours of the country had the same outline as now. At some points the gravel was quite green, from the debris of disintegrated basalts and greenstones, which have much the appearance as if they had come from near M'Duff's cave, and if so, the current must have flowed from the west. This whole deposit seems to be a portion of the higher level sands and gravels which are found over the surface of the country,

and on which the mansion-house of Cairnie, for example, and the town of Colinsburgh rest.

At this point there occurs the same break in the succession as in the above section. Though not as prominently shown, it is really present.

4. The Arctic shell-clay. The upper portion consists of sand, for the most part unstratified; but at two points it was observed to be separated into layers by partings of minutely triturated coal shale. This sand interpenetrates and passes into the clay, which corresponds in structure exactly with that of the shore-section, and, like it, contains shells in exactly the same state of preservation.

5. The boulder-clay in three stages.

The highest portion to the east was black, with almost a bluish tinge. At a somewhat lower stage it shows a brown colour, while lowest is the basement bed full of shivers from the shaly rock on which it rests. The boulders are numerous and large, some of them two yards in diameter and striated. Boulders, indeed, are found all through these beds up to the top of the sand forming the upper portion of No. 4, but they are much the most numerous in No. 5.

Transverse Section, Elie.

It seemed desirable to examine these deposits on a line, as nearly as possible, at right angles to the previous sections; and with this view I took the banks of a small stream, which falls into the sea a little to the west of Kincraig—the Cocklemill Burn. A mass of trap, *t*, springs up close to high-water mark, and

Section IV.

Fig. 4.

behind it there lies a small patch of boulder-clay, *b*. Beginning at the surface, however, and taking the beds in order, we have—

1. Blown sand rising into dunes.

2. The shingle, and shells, and sands of the so-called raised beach. It runs about half a mile up the banks of the stream, and rises at certain points 18 or 20 feet above high-water mark. The shells are the common species of the Frith, and as fresh as those now lying on the shore.

3. The third deposit consists of sands and clays, arranged in contorted layers. Its junction with No. 2 is not seen, but that it is older, and, therefore, underlying, is proved by the much worse preservation of the shells. When first laid open, a

little way above the foot-bridge, the base is found to consist of fine blue clay, passing upwards into a series of numerous sandy and clayey layers, arranged alternately. One of these layers of clay, near the summit, deserves particular attention, from the shells which it contains, and to which I shall afterwards refer (fig. 6.)

4. The submerged forest of Largo Bay. Close to the Shooter's point, where this section strikes the shore, it is found between high and low water mark, and is seen passing out seawards. As it is an outlier, and not in contact with the preceding beds, there may be some difficulty in fixing its position in the order of succession; but there seems good ground for placing it, as we have done, immediately beneath the bed No. 3. The enclosed organisms show that it is the same with the peat, No. 3 of Section II.; and that it is evidently identical with bed No. 2 of Section III.; and we are thus able to show, approximately at least, its true stratigraphical position.

It will be observed that we learn from Section II., that it is older than the so-called raised beach, and newer than the line of non-conformity—the break formerly referred to—while Section III. shows that it is newer than the high-level gravels and sands of the district. I was anxious to ascertain on what it rested, as found on the shore, for the purpose of comparison. Mr Howie, of Largo, who has paid much attention to this deposit, kindly met me on the spot. The peat was dug through, and the underlying strata laid open, when we found the following result.

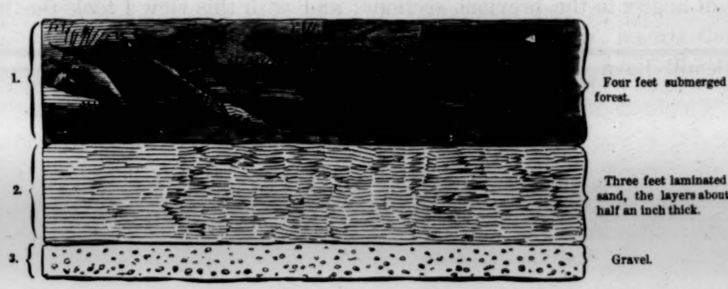


Fig. 5.

It will be observed how closely this corresponds with Section III. The sandy and gravelly layers agree with the deposit which there underlies the peat (2), and seem to indicate that this peat immediately succeeds the high-level gravels of the district, coming in directly beneath the deposit No. 3 of Section IV.

Having thus seen in detail the nature of these separate sections, it may be well now to combine them in one view, and glance at the leading events in their order. My object is to show at what point of our recent geological history the Arctic shell-clay comes in; and perhaps this will be most clearly seen if we

begin at the surface, and work our way down step by step to the lower deposits, noticing as we go the evidence as to climate, and the relative height of sea and land. Immediately beneath the surface, then, we find the first stage, comprising the two newest deposits.

I.—*The Blown Sand and Raised Beaches.*

Of the blown sand, by far the best display is at the Elie railway station, where, from the highest point of the synclinal down to the base of the deposit there must be at least 20 to 30 feet of perpendicular depth. The highest bed of the enclosed peat is about 6 feet thick. The growth of this peat at its different levels, and the accumulation of this sand, obviously show that the lower portion of it must be of considerable antiquity. The great feature of the deposit is the profusion of land and fresh-water shells in the peat. I examined the lowest bed with some interest, to ascertain whether any of the species were extinct, but found only the following:—

Succinea putris.	Helix nemoralis.
Limneus pereger.	fulva.
Zua lubrica.	fusca.
Pisidium pulchellum.	pulchella.
Cyclas cornea.	Pupa muscorum.
Carychium minimum.	Planorbis marginatus.

These are all recent, and most of them have been actually found by Dr M'Bain living near Elie. The only thing to be observed is, that the immense numbers of these shells, found in the peat, seem to show that there formerly prevailed some peculiarly favourable conditions for the development of this form of life. At the same time, it is clear that the climate must have been much the same as now, for the species are identical.

Along with this deposit may be classed the so-called raised beach. The great display of it is seen in Section IV., where it is cut open for half a mile across, and is at some points 18 feet in depth, rising to that extent above high-water mark. The materials, consisting of shingle, sand, and shells, have been thrown up by the sea, and apparently at different times, and in a confused way. The shells are all of species now common, and so well preserved, that there is really no reason to think the deposit is any older than some of the oldest portions of the blown sand seen in Section III. The formation of the two may quite well have gone on together—the one inland, and the other at the sea-shore.

There has been a good deal of discussion as to whether this deposit indicates a rise in the land. The point, I must confess, seems still doubtful. The sea in certain states has, we know, the power of throwing up shingle and shells to a considerable height;* and when species of dead shells are found, as here, con-

* See this view well stated by the Rev. W. Wood, in his work on the East Neuk of Fife, p. 320.

fusedly cast together with other materials, I feel as if the evidence, had it stood alone, would have been far from decisive. These two deposits, then, taken together, form the first and most recent stage.

II.—*The Sands and Clays along the Banks of the Stream.*

This second stage is seen only in Section IV. Along the course of the stream are the *present* banks, corresponding with the first stage. But farther back, on either hand, and rising higher, is the deposit in question, which represents a previous state of things. A little above the foot-bridge the stream has cut into it, and laid open the following section already referred to. In this the

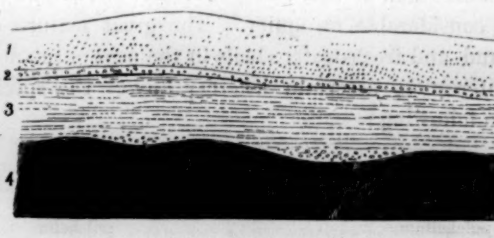


Fig. 6.

bed No. 2 deserves particular attention, as furnishing good evidence of a rise in the bed of the sea. It consists of a layer of fine clay, about a foot in thickness, running all along the section, immediately under the blown sand, and contains a good many shells of a single species, the *Scrobicularia piperata*. It may be termed the *Scrobicularia* bed. Now, the remarkable thing is, that almost the whole of these shells are in an erect position, with the siphonal end uppermost—showing, as every conchologist knows, they are at home just in the position in which they were when alive. The habits of the shell are well enough known. It burrows in clay at the mouths of rivers, sending up its long siphons to draw a continuous stream of water into its gills, so that in this bed the shells are in the position in which they lived. The habitat of the shell also is peculiar and well marked. It enters the mouths of tidal rivers, but never goes out of the reach of the tide. It needs to have not only the fresh water, but the salt also, flowing over the ends of its siphons. Now, the tide runs up the stream past this section, and the point to be specially noted is that the *Scrobicularia* bed is not less than 14 feet above medium high-water mark. The conclusion seems irresistible, that at the time these shells lived the land was not less than 14 feet deeper in the sea than now. There is something very remarkable, also, in the way in which this evidence harmonises with what is found elsewhere round the shores of the Firth. Between Stirling and Bridge of Allan, Dr M'Bain informs me he obtained from the brickfield specimens of this same *Scrobicularia*, about 15 feet above high-water mark. At various

points, also, above Stirling, specimens of whales have been found imbedded in the same clay; and in two cases it is recorded that there lay beside them rude spear-heads of deer-horn, with which they seem to have been killed. Curiously enough, at Portobello a bed of the same *Scrobicularia* was found in the brickfield; and it lay, as Dr M'BAIN informs me, at the same height of 15 feet above high-water mark. Thus all round the Firth the evidence points to the same conclusion. The land was at least 14 or 15 feet deeper in the water than now. The sea ran above Stirling, where now there is solid ground, and if the evidence as to those deer-horn weapons may be relied on, it would seem that at the time these shells near Elie were sending up their siphons to be swept by the waters of each returning tide, our forefathers, with their rude harpoons, were hunting the whale, where now only the green fields of the farmer are to be found.

Another point not less clear is, that the climate must have been much the same as now. In the above section (fig. 6), the *Scrobicularia* bed is underlaid by two yards of alternating sandy and clayey layers; and among these, near the base, the following additional species of shells were found:—

Tellina solidula.	Mytilus edulis.
Turtonia minuta.	Littorina littorea.
Cardium edule.	Montacuta ferruginosa.
Rissoa ulvæ.	

This group is at once decisive as to the question of climate. Take, for example, the *Rissoa*, which occurs in great abundance. It lives between tide marks, and is exposed to all atmospheric changes. It seems to have been as abundant when these beds were laid down as now. The climate then must have been much the same as the present. Passing this stage, we come to the third in the descending series, viz.:—

III.—*The Submerged Forest.*

This we have seen, in Section II., in the act of passing seaward; it is seen again in Section III, as a thin bed of peat; but it is especially in Largo Bay that it comes out in full force. Somewhat to the west of the point where Section IV. passes, it is 4 feet in depth (fig. 5), showing a great peaty mass, in which willow and hazel, and especially hazel nuts, were found, with other seeds, mosses, and especially the abundant remains of *Arundo phragmites*. Though now lying deep under every tide, yet no single marine organism enters into the bed itself. It sweeps for miles all round Largo Bay, and Dr FLEMING has described trunks of trees standing in it rooted in the soil beneath. At Aberdeen, in the Tay, at various points in the eastern and southern coasts of England, and again across the Channel and in Jersey and Guernsey, the same deposits are found, and it seems really impossible to resist the inference, that they indicate a time when the land stood higher above

the sea than now. Here, in Largo Bay, a land surface, with rooted stumps of trees, is seen passing out seaward at low-water mark. It may well have been that the Britain of that time stood so high above the water as to have been connected with the Continent. The only thing to be noted is, that if this were the period of such connection, it must have been subsequent to the glacial epoch. At the time of these submerged forests the *Arundo* grew abundantly, as now, in the marsh, the willow put forth its catkins, and the hazel ripened its nuts in the wood, and the whole flora, down even to the mosses, was the same as now. Thus far the climate must have resembled the present, with no trace of glacial cold.

IV.—*High-level Gravel and Sand.*

This has been described in noticing the details of Section III., where it forms bed No. 3. It should be distinguished from the second stage in this enumeration. The latter lies only along the course of the stream, and is a river deposit, with estuarine shells and no gravel, while this fourth stage consists of gravels and sands, utterly without fossils, and scattered at considerable heights all over the surface of the country. It was well displayed in the railway cutting. In one of the layers of rather coarse gravel there were angular patches of fine sand; and their presence could only be explained by supposing that some sandy stratum had been frozen, the current had gained access to it, broken it into fragments, which in a frozen solid state had been transported along with the gravel, and deposited where they now lie as angular masses of fine sand crumbling to the touch, but having their angles sharply defined. If this be the right explanation, then we have here the first indication of glacial cold in these deposits; and, perhaps, it points to the true origin of these masses of high level unfossiliferous stratified gravels and sands found over Scotland. They may be due to the floods attending the breaking up of the glacial epoch.

V.—*The Arctic Shell-Clay.*

Passing over the break in the series formerly referred to, we come to what is the chief object of this paper, the period of the Arctic shell-clay. When my attention was first called to the subject in 1862, no deposit of Arctic fossil shells had been found, either on the Forth or Tay. The first trace, indeed, had been detected by Dr Fleming, Dr M'Bain, and the late Mr Bryson, but only two or three specimens of the shells had been got, and they were held not to be indigenous. The seeming absence of this fossil fauna from our eastern friths had led some geologists of eminence to question the inferences drawn from those of the Clyde. The discovery of the Elie shells, however, was at once decisive. Through the obliging attention of Mr Carruthers of the British Museum, they were submitted to Dr Otto Törell of Lund, who has himself dredged extensively in the Arctic seas, is the author of an important work on the Shells of Spitzbergen, and is of

the highest authority in this special department. He had the goodness not only to name my specimens, but went with me to Elie to examine the shells as they lay in the clay. I have to acknowledge my obligations to him for much valuable information. I made no attempt at a complete collection of the shells. They are so friable, that much time and patience is required to extract them from the clay, and I was content with having obtained the following species, which seemed sufficient to determine the character of the deposit.

1.	Buccinium cyaneum,	Errol.
2.	Natica grœnlandica,	Elie, Errol.
3.	Turritella erosa (polaris),	Elie.
4.	Pecten grœnlandicus,	Elie, Errol.
5.	Crenella decussata,	Elie.
6.	nigra,	Errol.
7.	lævigata,	Elie, Errol.
8.	Leda (Yoldia) truncata,	Elie, Errol.
9.	minuta,	Elie, Errol.
10.	Yoldia hyperborea,	Errol.
11.	(*new species*),	Elie.
12.	Astarte compressa,	Elie.
13.	Nucula inflata,	Elie, Errol.
14.	Dacrydium vitreum,	Errol.
15.	Thracia myopsis,	Elie.
16.	(*new species*),*	Elie, Errol.
17.	Tellina proxima,	Elie.
18.	Saxicava rugosa,	Elie, Errol.
	Fusus *sp.*,	Elie.
	Cylichnia ? *sp.*,	Elie.
	Pleurotoma *sp.*,	Errol.
	Axinus ? *sp.*,	Errol.
	Univalve, unknown,	Errol.

In looking over this list, the first thing to observe is, that it contains no single species which does not now live in the Arctic seas. In this respect the deposit differs markedly from the Clyde beds, where a certain proportion of the species cannot bear the cold of the northern ocean, and are yet lying side by side with boreal species. It becomes a somewhat complicated problem, and would require careful consideration, how such incompatible results are to be explained. But at Elie and Errol the problem is simple. The whole species lying in the deposit are now living in the Polar seas. As to the evidence thus furnished in regard to climate, they may be divided into three classes:—

1. There are some which tell nothing, either for or against a period of Arctic cold. The *Crenella nigra*, for example, reaches its southern limits in the Scottish seas, where it used to live among the oysters of the Frith of Forth. It ranges north to Spitzbergen, and the remarkable thing is, that it does not seem to be affected by the change of climate. The discovery of such shells could furnish no evidence either way.

2. There is a second class which can live indifferently in the British or

* Undescribed, from the Northern Shores of Spitzbergen.

Arctic seas, but they vary markedly with the degree of cold. Of this kind are the *Natica grœnlandica* and the *Saxicava rugosa*. The *Saxicava*, for example, is at home everywhere, from the Polar seas to the Canary Islands; but under an Arctic climate it thickens its shell, and attains a size quite unlike what it does with us.

To show how completely the specimens from this deposit are of the northern type, I place side by side No. 1, the outline of the shell of the full present British size, as given by FORBES and HANLEY, and No. 2 and 3, the same shell, as found in the Elie clay.

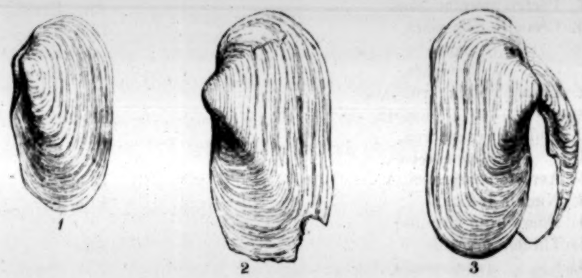


Fig. 7.

The evidence of the Arctic climate is here very marked.

3. The third and by far the most numerous class consists of shells, which require the climate of the Polar seas, and really can live nowhere else.

The two marking and characteristic shells of the deposit are the *Leda truncata* and the *Pecten grœnlandicus*. Both reach their southern limits on the coast of Norway. Both shells were found by Dr TÖRELL living together in the clayey sea-bottom, in front of the great glacier of Spitzbergen, just as they now lie together in this Elie and Errol clay. Another point of importance is the comparative abundance of this *Leda*. It is one of the most abundant shells now at Spitzbergen, and it is beyond all comparison the most abundant shell in this deposit, holding the same place that *Tellina proxima* does in the Clyde beds. The *Tellina*, on the contrary, is rare at Elie and Errol. I obtained only two or three specimens. The details as to the distribution of other species are not less decisive.

Buccinium cyaneum is found in Greenland, &c.

Turritella erosa.—Of this I got only one specimen, and it was not quite perfect. Dr T. says of it, " Almost certainly the same species, yet cannot be positively asserted." It lives in the seas of Greenland.

Thracia myopsis.—Iceland, Greenland, Spitzbergen, in from 60 to 200 fathoms water.

Nucula inflata.—Spitzbergen, 5 to 150 fathoms.

Crenella lævigata.—At my first visit to Elie I got a few specimens, which Dr

Törell marked as "*C. lævigata* most probably, but much injured." Afterwards at Errol the workmen at one place threw them out by spadefuls, well preserved; and on showing them to Dr Törell, he pronounced them most characteristic examples of the species. He had found the shell very abundantly at Spitzbergen, where it forms a great part of the food of the walrus, the large tusks of the animal being used to dig it out of the clay of the sea-bottom.

The new species of the *Yoldia* (No. 11 of the above list) also deserves notice. It is an unpublished species, dredged by Dr T. at Spitzbergen, in 80 degees of north latitude, and the next place where it turns up is this Elie clay. Not less interesting is the *Dacrydium vitreum*, a small shell, of which he had dredged a very few specimens in Spitzbergen, and on these had founded the new genus of Dacrydium.* The next place where it is met with is the clay at Errol, the deposit thus yielding some even of the rarer species of Arctic shells, whose proper habitat is at Spitzbergen, within about ten degrees of the Pole, and close to the limits of perpetual ice.

From all the evidence thus obtained, the conclusion will be held to be irresistible, that at the time these shells lived this country lay under the most rigorous Arctic climate. Like Greenland or Spitzbergen now, the Scotland of that day was wrapped in snow—a land of glaciers and icebergs.

In addition to the shells, I obtained from the Errol brickfield portions of the skeleton of a seal, in regard to which Dr M'Bain has favoured me with the following note:—"The vertebræ enclosed in the Errol brick-clay are those of a young seal, and, from a comparison of the few detached vertebræ, they appear to me to belong to *Calocephalus vitulinus*. This could only be satisfactorily determined, however, by the discovery of the skull. The loose bones are an atlas the sixth and seventh cervical, and the first dorsal vertebræ. The others are portions of ribs, one with the head and neck, probably the twelfth, and from their size appear to belong to the same individual."

In regard to the relative levels of land and water, it is plain that at the period of this deposit the country must have been sunk far deeper in the sea than now. The Errol deposit at this moment lies more than 40 feet above high-water mark. No one can look on the shells, as they lie in the clay, without seeing that they are at home—not washed up, but in what was once the sea-bottom, where they lived and died. The old and young are so mixed together—the most tender portions of the epidermis, &c., are so preserved—that this cannot be questioned. Lying, therefore, now 40 feet above high water, a rise to that extent at least is undeniable. But there is much more than that. None of the shells are littoral species; and the group would, as a whole, require, when alive, a considerable depth of water over them. Much attention has been paid to the range of depth through which the different species live. If this were exactly ascertained, we should

* Spitsbergens Mollusker I. p. 19.

only have to consider at what point of depth all the above species could meet. Some of them have a wide range, like the *Crenella lævigata*, for example, which is found at all parts of the sea-bottom, from 5 to 200 fathoms. Others, like the *Natica grœnlandica*, have only been got from deep water. On the whole, however, our information as to the range of shells will hardly allow us to do more than form an approximate opinion. The species on which the conclusion must mainly turn is the *Leda truncata*, which is beyond all comparison the shell of the deposit. At Spitzbergen, &c., it is described as ranging from 5 to 30 fathoms; and between these two points, therefore, the degree of depth must lie. But, looking to the shells with which it is associated, it must evidently lie somewhere about the lower portion of its range. The *Thracia myopsis*, for example, is found at 60 fathoms, and ranges from that down to 200. The new species of *Yoldia* was dredged only in 100 fathoms. The presence of such shells would seem to indicate that we are to fix the point somewhere towards the lower range of the *Leda*, say 160 feet. This is the deposit which is now lying more than 40 feet above the sea; and we can hardly be wrong in concluding, that at the time these shells lived the level of the land was at least 150 to 200 feet lower than now.

V.—*Boulder-Clay*.

The lowest stage of these deposits is that of the boulder-clay, which, both at Errol and Elie, is found lying beneath the Arctic shell-clay, and resting immediately on the rock. It would seem that this lowest deposit, so long an enigma, has at last yielded up its secret,—that it is a land deposit, formed at the period when Scotland, like Spitzbergen, lay beneath an immense covering of ice, which wrapped the whole face of the country, hill and dale. Underneath such a covering, possibly thousands of feet in thickness, the rocks would be ground down, and the boulder-clay formed. And thus the absence of fossils is accounted for, inasmuch as none of our usual forms of life could exist beneath such an ice-sheet. And thus we see also how the clay is so peculiarly hard and untractable. This hardness is very strikingly noticed wherever the true boulder-clay is dug into, as at the mantrap, near Leith, or in the neighbourhood of Greenock, where, in the excavations at the harbour, I found them blasting it with gunpowder. It would seem as if the pressure of immense ice-masses passing over it had compacted it into such hardness. This feature was very marked in the railway cutting at Elie.

As to the relations of the boulder-clay and the Arctic shell-clay, there are two questions to be considered :—

1. Is the boulder-clay an antecedent formation, and did the Arctic shell-clay come to be formed only after the formation of the boulder-clay had ceased?

Both at Elie and Errol the shell-clay rests on the other; and it is, of course, plain, that the particular portion of the boulder-clay there present is older than that portion of the shell-clay. But we must take care how we extend this

inference to the two deposits as a whole. If we endeavour to picture the time when the shell-clay was laid down, it will be plain that all through it there must have gone on the process of the formation of the boulder-clay. We can tell to a certainty that the greater portion of Scotland was above water when the Elie and Errol shells lived, for the characteristic species, the *Leda truncata*, does not live in more than 50 fathoms of depth. A great part of the country, therefore, stood out of the water, and all over the higher grounds the great ice-sheet was doing its work, in forming the boulder-clay. When the shells lived in the Elie clay, the glaciers were scouring the valleys of the Pentlands, and rising over the sides of Arthur's seat. When the current swept the sea-beds of Errol, the ice-sheet was grinding the flanks of Benvoirlich, and ploughing its way down the solitudes of Glenartney. This is the story of that epoch, which admits of no denial. There was land, and there was sea, and the two formations were cotemporaneous—the Arctic shell-clay under the sea, and the boulder-clay on the land. Take any portion of the shell-clay, and you would probably find that if the whole facts could be brought out, some portion of the boulder-clay was prior, some cotemporaneous, some posterior to it. But as a whole, the two were simultaneous. As surely as there was land and sea, so the land and the sea formations ran on side by side, just as the two processes may be seen this day going on side by side at Greenland or Spitzbergen.

2. Then a second question is, whether the boulder-clay is or is not fossiliferous? Some have been inclined to view it as marking a time of dreariness and desolation, almost as azoic, a period of death, so devoid is it of fossils. This, I believe, is a mistake. If we use the term boulder-clay in a mineralogical sense for a certain clay in a certain mechanical state, then, as a whole, it is unfossiliferous, and we have seen why it must be so. But if we use it in a geological sense as representing a particular epoch, then the fossils of this Arctic shell-clay are the fossils of the boulder-clay. In them we have the marine life of the boulder-clay period. The fossil fish of the Old Red Sandstone belong to their own epoch, none the less that the bed in which they are found may not be sandstone and not red, and in this sense the fossils of the Arctic clay might be termed boulder-clay fossils, as representing the marine life of that epoch of which the boulder-clay was the great feature. Having thus glanced at the details of the different deposits, let us now advert to the

General Results.

The only safe rule in classifying the beds of any geological epoch is by the enclosed fossils. Were we to take the boulder-clays, brick-clays, and gravels, and try to work out the succession by the mineral and mechanical structure, we should be groping in the dark. Dr Fleming showed that there are earlier and later boulder-clays; and this is quite plain, when we contrast that of Errol, under-

lying the shell-clay, with that of Caithness, as investigated by Mr PEACH. The same thing is true of the brick-clays, and we should thus get into all kinds of confusion did we attempt to make out the series by attending merely to the structure of the beds. It is by examining their fossils or their stratigraphical position, as compared with other fossiliferous deposits, that we shall be safe in our inferences. Proceeding in this way, the whole deposits referred to in this paper may be arranged in three divisions.

I. The Elie and Errol shell-clay, with its underlying and (in the sense explained above) contemporaneous boulder-clay. All over the Forth and Tay districts this is the lowest and oldest portion representing the period of the Arctic cold.

II. There is an intermediate series of deposits. In the above sections described in this paper these beds are awanting, represented only by the line of unconformity formerly referred to as making a break in the series. At a single leap we pass from the rigour of Arctic cold to the present climate. This chasm, however, may quite well be filled up from other parts of the country where we pass through a whole series, in which a less and less degree of cold is gradually indicated. About the centre of this scale would seem to lie the Clyde beds, described by Mr SMITH of Jordanhill, having beneath them those of Aberdeen, so well investigated by Mr JAMESON of Ellon, and above them those of Fort William and Caithness. Step by step we can trace the passing away of the Arctic cold. These intermediate deposits are wholly wanting in the above sections, unless it may be the portion of high-level unfossiliferous gravels (Section III.), which show, as we saw, some trace of glacial cold.

III. The group of deposits representing the present climate.

It will not, I trust, seem presumptuous if I suggest that geologists may yet find in these Elie and Errol deposits the starting-point for a more rigorous classification of our superficial beds throughout central Scotland. In studying the geographical distribution of northern shells as they at present exist, it is well known that conchologists have recognised two great provinces—the Arctic, or most northern, and the Boreal, or sub-arctic, the less northern. Now, as this division holds good for the conchologist in separating the two groups in regard to space, so the geologist may find it hold good in separating two groups of deposits in regard to time. We have in these superficial strata first an Arctic time, with its own set of deposits, when the climate is that of the Polar regions, characterised by shells strictly Arctic, and next a sub-arctic time, also with its special beds, characterised by the group of Boreal shells. The type of the one is the *Leda truncata*, found everywhere in the lower and colder deposit; it represents the Arctic province. The type of the other is the *Tellina proxima*, found everywhere in the Clyde beds in countless numbers.* It is properly a Boreal shell, repre-

* It occurs but very scantily at Elie.

senting the Boreal province and the sub-arctic deposits. The one may shade into the other, just as the provinces of the conchologist do; but rightly viewed they may serve as great landmarks, enabling us to classify and so to advance our knowledge. With this explanation I would submit the following diagram, as embodying a general view of the sequence of deposits through the whole of the sections referred to in the preceding pages. It is not meant that the formation of each underlying deposit was finished before the overlying bed began to be laid down, at least in those cases where such deposits are found in separate localities. Just as the Arctic shell-clay was (in the way explained above) partly cotemporaneous with the boulder-clay, so the blown sand, No. 1, may have been cotemporaneous in part with the raised beach, No. 2, and even with the river deposit, No. 3. It is often impossible to establish the sequence in a rigorous way, so as to exclude the idea of cotemporaneous formation in greater or lesser degree among these superficial beds each in its own separate locality. But allowing for this, the following will, I believe, be found to approximate closely the sequence of the deposits:—

Diagram showing the order of the Superficial Deposits.

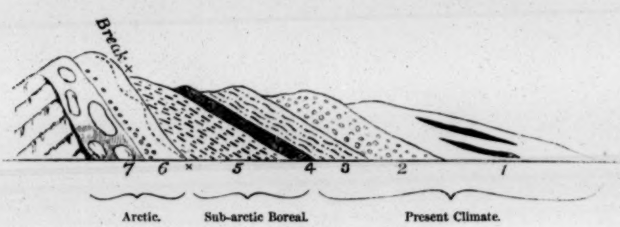


Fig. 8.

1. Blown sand, with peat full of land shells.
2. Raised beach.
3. River sands and clays—*Scrobicularia* beds.
4. Underlying peat—submerged forests—no shells yet found.
5. High-level gravels.
× Break in series of deposits representing the time of the Clyde beds, &c.
6. Arctic shell-clay of Elie and Errol.
7. Boulder-clay.

Fig. 1

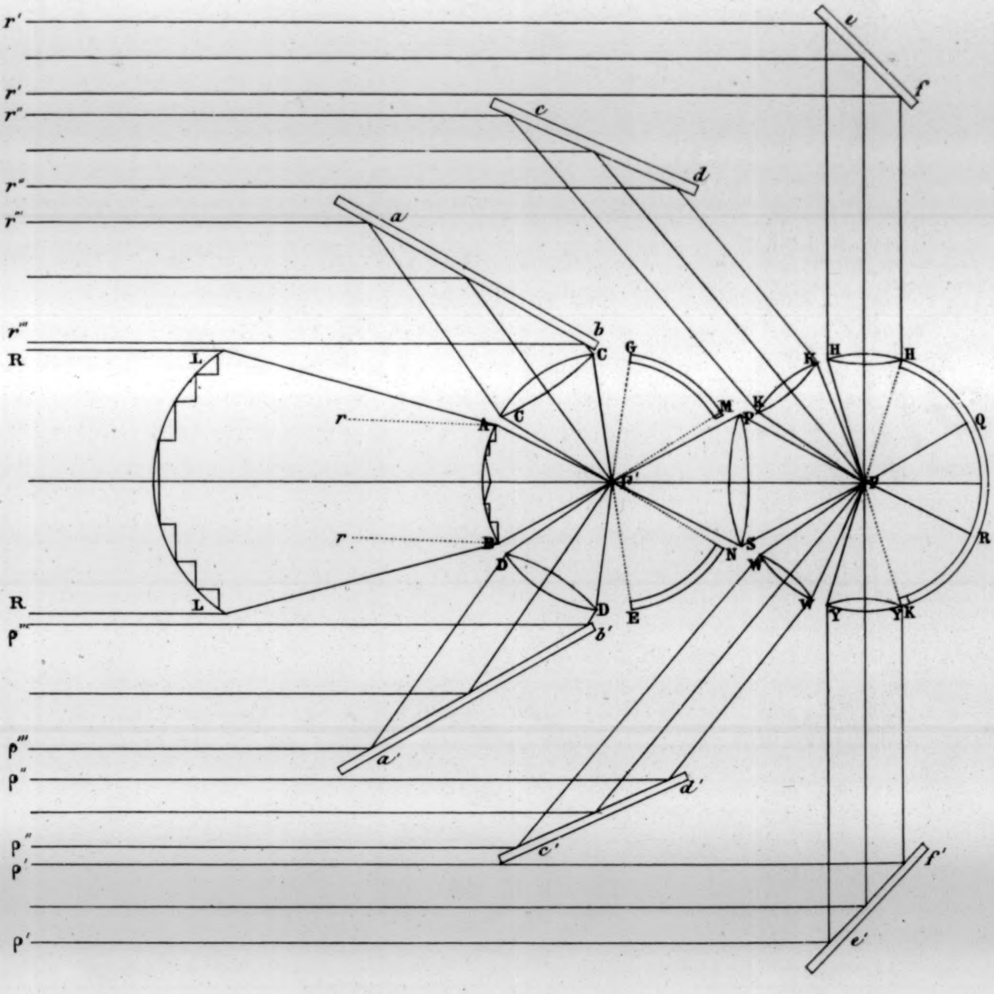
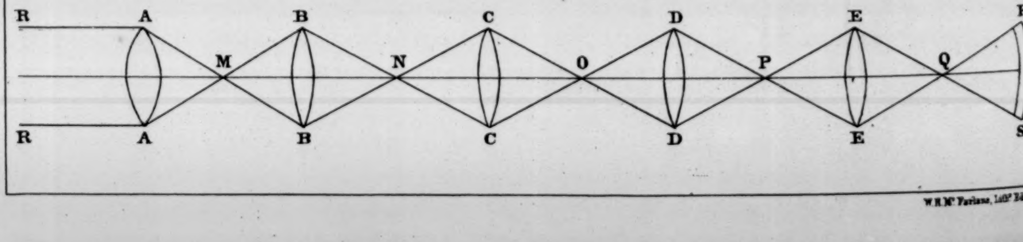


Fig. 2.



W.H.Mc Farlane, Lithr. Edinr.

XLI.—*Description of a Double Holophote Apparatus for Lighthouses, and of a Method of Introducing the Electric or other Lights.* By Sir DAVID BREWSTER, K.H., D.C.L., F.R.S. (Plate XXXVIII.)

(Read 29th April 1867.)

In the year 1812 I described an apparatus, by which the light of the sun, or of any luminous body concentrated in the focus of an improved lens, could be returned by reflection from a spherical mirror into the same focus, thus increasing the light and heat produced by the cone of refracted rays. The apparatus contained also a double system of lenses and plain mirrors, by which additional beams of light could be concentrated in the same focus.*

In 1827 I described the very same apparatus as applied in the dioptric system of lights, *the whole of the light* which issues from a lamp being thrown into one wide and parallel beam, constituting what has been called a *holophote*, now in use in every part of the world.†

This apparatus consists of *three* different parts:—

1*st*, Of a lens which refracts into a parallel beam of light a cone of rays, which has for its base the surface of the lens, and for its apex the source of light.

This use of a common lens was made in the lighthouse in the Isle of Portland in 1789.

2*d*, Of a spherical mirror placed behind the flame, which throws into the parallel beam a similar cone of rays, whose base is the mirror, and whose apex is the flame.

3*d*, Of two systems of lenses and plane mirrors, by which the cones of rays, which would otherwise be lost, are thrown into cylindrical beams, which widen the principal parallel beam.

If the single holophote has been found so useful in lighthouse illumination, a double or even a triple holophote, in which all the lenses and mirrors may be reduced in size, must, in particular circumstances, have a peculiar value.

A double holophote, in which the light of two flames is condensed into a wide beam, is shown in Plate XXXVIII. Fig. 1, where F, F′ are the two flames surrounded by lenses either of one or more pieces, and by plane and spherical reflectors, which may be made of speculum metal, of prisms, or of glass silvered behind or before. The cone of rays FPS, issuing from the flame F, is converged

* Edinburgh Encyclopædia, Art. BURNING INSTRUMENTS.

† Edinburgh Transactions, 1827, vol. xi. pp. 55, 56.

to the focus F′, from which it diverges, and falling upon the lens AB, is refracted by it, and the lens LL into the parallel beam RR, or, if we use only the lens AB, into the parallel beam *rr*.

The cone of rays FQR, which radiates from F, is thrown back to F by the part QR of the spherical mirror HQRK, and falling upon the lens PSP, is refracted like the direct cone FPS into the parallel beam RR or *rr*.

The two cones of rays FKK, FHH, intercepted by the lenses KK, HH, are refracted into parallel beams, which, falling upon the plane reflectors *cd*, *ef*, are reflected into the parallel beams $r''r''$, $r'r'$. In like manner, the cones of rays FWW, FYY are refracted by the lenses WW, YY into parallel beams, which, falling upon the plane reflectors $c'd'$, $e'f'$, are thrown into the parallel beams $\rho''\rho''$, $\rho'\rho'$.

The cone of rays FRY, falling upon the part RY of the spherical mirror HQRY, is returned through the flame F, from which it diverges, and falls as a second cone upon the lens KK, which throws it into a parallel beam, which is reflected into the beam $r''r''$ by the reflector *cd*. In like manner the cone of rays FHQ is, by means of the lens WW, and reflector $c'd'$, thrown into the beam $\rho'\rho'$.

The eight cones of rays which issue from the flame F, being the whole of its light, thus passes into the wide compound beam $r'\rho'$, consisting of the principal beam RR, and the four beams $r'r'$, $r''r''$, $\rho'\rho'$, $\rho''\rho''$. The beams $r'r'$ and $\rho'\rho'$ are each composed of *one* cone of rays FHH, FRY, and the beams $r''r''$, $\rho''\rho'$ of two cones of rays, the one direct, and the other reflected.

If we now place a second flame at F′, and surround it with the lenses and reflectors shown in the figure, it will become a second holophote. The cone of rays F′AB, refracted by the lenses AB, LL, will add its light to the central beam RR, and the opposite cone F′PS, refracted by the lens PS′ to F, from which it falls upon the mirror QR, will be reflected through the flame F′, from which it diverges and enters into the beam RR, like the direct cone F′AB.

The cone F′CC, refracted by the lens CC, is reflected by the plane mirror *ab*, so as to form a new beam $r'''r'''$, filling up the space between the beam $r''r''$, and the central beam RR. In like manner the cone F′DD is made to fill up the space between $\rho''\rho''$ and RR, by means of the lens DD and the reflector $a'b'$.

The cone of rays F′GM, intercepted by the spherical reflector GM, as reflected through the focus F′, and by means of the lens DD and the reflector $a'b'$, is thrown into the beam $\rho'''\rho''$. In like manner the cone of rays F′EN, reflected by the spherical mirror EN to the flame F, will pass into the beam $r'''r'''$, by the refraction of the lens CC and the reflector *ab*.

If it is desirable to have, for any special purpose, a more intense and a wider beam, we have only to substitute a lens in place of the mirror QR, and introduce through it the light of one or more holophotes, the cones of rays passing through the lens at QR, adding to the intensity of the central beam RR, the

other cones of rays which issue from the new flames producing hollow cylindrical beams, surrounding those shown in the figure.

If we wish to have a beam of great intensity, and of equal brightness throughout, the holophote principle is inapplicable; but in dispensing with the secondary lenses and the lateral reflectors, we obtain a less complex and more manageable apparatus, to which the name of *Cratophote* may be given, from the great intensity of the light which it is capable of producing.

This instrument is shown in Fig. 2, where M, N, O, P, Q are five flames, the light of which we wish to concentrate into one beam RR, MN, a concave mirror, and AA, BB, CC, DD, EE, &c., five, or any other number, lenses. The lenses BB, CC, DD, EE have the same focal length, but the focal length of AA is only one-half that of the rest. The light of the *first* flame M, or the cone of rays MAA, is refracted into the parallel beam RR. The light of the *second* flame N, or the cone of rays NBB, is refracted by the lens BB into its conjugate focus M, and, passing through it, falls upon the first lens AB, and is thrown by it into the beam RR.

In like manner all the other flames or cones of rays, OCC, PDD, and QEE, are converged by the lenses which intercept them to their conjugate foci, and finally enter the beam RR, into which the light of the five cones of rays is concentrated.

By placing a concave mirror RS in the instrument, so as to reflect the cone of rays QRS to Q, they will pass through all the other lenses into the beam RR. In like manner the other cones of rays PEE, ODD, NCC, MBB will be converged to the conjugate foci of the lenses which intercept them, and fall upon the mirror RS, which will send them back by the same process into the beam RR.

The intensity of the beam RR may be increased by diminishing the diameter and the focal length of the lens AB, which, when placed nearer N, will refract the cone of rays MAA into a beam of the same diameter as the lens.

The intense beam of light produced by this apparatus is obviously very different in its character from the compound beam obtained from the single or the double holophote. In the single holophote the beam consists of the central beam RR, Fig. 1, and of the hollow cylindrical beam $\rho'''\rho'''$, which, being less intense, will not be seen at such a distance as RR. In the double holophote the beam consists of the central beam RR, and of *three* cylindrical beams, each of which has a different intensity, and will, therefore, cease to be seen at different distances. It might, therefore, be desirable that each part of a lighthouse beam should have the same intensity, and, consequently, the same penetrating power as in the catoptric system, where the beam is produced by several parabolic reflectors of the same size, placed close to each other. In the dioptric system we can only approximate to this, and that very imperfectly, by combining several small holophotes, and thus producing a wide beam, in which we have several central beams,

each of which is surrounded by its feeble cylindrical ones. In this constructicn we might dispense with the use of large built-up lenses, and employ small lenses of flint-glass, which can now be manufactured at a moderate expense, and of great excellence.

The suggestion to introduce the Drummond or the electric and magnesian lights as *permanent* lights in our lighthouses is not likely to be adopted. Oil and gas light, concentrated optically into a powerful beam, has a sufficiently penetrating power in ordinary states of the weather, and more brilliant lights are required only in fogs and states of the weather when feebler lights cannot be seen.

In my paper on the Illumination of Lighthouses, in our *Transactions* for 1827, I have shown in Plate III. Fig. 1, two methods of occasionally introducing the electric or other light into the central beam of a holophote, without interfering with its ordinary action.* If we consider the front holophote in fig. 1 to be the ordinary dioptric light, the second holophote, with the electric or other light at F, might be used in place of the simpler and less effective one above referred to.

* The method of doing this by lenses, whose conjugate focus is F', Fig. 1, or by an ellipsoidal mirror, one of whose foci is F, when the spherical mirror (MN, Plate III. Fig. 1, above referred to) is removed, is distinctly shown; but the note describing it was accidentally omitted by the printer.

XLII.—*On a Lower Limit to the Power exerted in the Function of Parturition.* By J. Matthews Duncan, M.D., &c. &c.

(Read 29th April 1867.)

The dynamics of natural labour have been the field of very little successful study or investigation. The object of the present paper is to make a contribution to this subject. I purpose to show what amount of pressure per square inch is sustained by the ovum in the easiest class of natural labours, and thence to estimate the propelling power exerted in such cases.

It is well known that natural births are ever and anon occurring, in which the ovum is expelled whole, the membranes containing the liquor amnii continuing entire. Into this category many more cases would enter, were it not a generally-followed rule for the attendant to rupture the bag should it advance entire as far as the external parts. Again, as Dr Poppel has pointed out, the attentive observer of a series of easy natural labours has no difficulty in arriving at the conclusion, that in not a few cases the same force which ruptures the bag of membranes is able to, and actually does, complete the delivery.

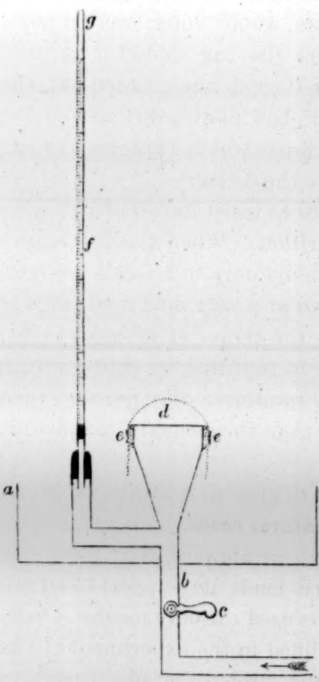
In all such cases, the strength of the membranes to resist impending rupture measures the force exerted in the process of parturition. When the bag is produced without laceration, its strength exceeds, certainly only to a small amount, the power of the labour. When the bag is ruptured at a very advanced stage of labour, as not rarely happens, its strength exceeds the power of labour exerted up till the time of its rupture. When the bag is ruptured by pains, which, without increasing in strength, rapidly and easily terminate the process, then the power of labour is probably only a little greater than the estimate, founded on the strength of the membranes, would indicate.

The strength of the membranes is thus shown to give us a means of ascertaining the power of labour in the easiest class of natural cases.

It might be suggested, that, in cases of persistent membranes, they were specially and unnaturally strong. My own experience lends no support to such a notion. Besides, so far as I know, no obstetrician has used the only means of verifying such a supposition—means such as are exemplified in the experiments to be hereafter related. Obstetricians have judged of the strength of membranes to resist a bursting force by their united thickness, or other less definite qualities, which form no criterion. It is not uncommon to read of the bag being strengthened by decidua; and that such thickening may be a source of strength is a common opinion; but as the decidua is far weaker and less extensible than the other membranes, the opinion is merely a natural delusion.

Experiments at once show that thickness of the membranes is no indication of strength. They also at once show that, for the special purposes of this paper, the amniotic membrane, being the strongest, alone requires to be observed. Long before the amnion is burst, the decidua and chorion have generally given way, and ceased to support the persistent amniotic membrane. The decidual membrane generally gives way first, under a bursting pressure applied to all three membranes. It sometimes does so with a sound as of a gentle fillip. Occasionally it bursts simultaneously with the chorion; and occasionally all three membranes burst at once. The decidua has been found, in the experiments, to burst at a tension of ·35 lb. per linear inch, corresponding under the circumstances supposed to exist in actual labour to a forward pressure of nearly 5 lbs.

As a general statement, it may be said that the chorion behaves like the decidua. It is of more uniform strength than the decidual membrane, and is only a little stronger, the average tensile strength being ·62 lbs. per linear inch, corresponding to a propelling power in labour of nearly 9 lbs. [In taking these averages, experiment 25 is omitted, because its exceptional value indicates almost certainly a mistake.]



The strength of the fœtal membranes lies in the innermost sac, in the amniotic membrane, which appears the thinnest and most delicate of all. To try the strength of it, as well as of the others, I made numerous experiments in the following manner:—They were all performed in the laboratory, and with the apparatus, of Professor TAIT, to whose knowledge and skill I am indebted for their value and accuracy. The apparatus used was connected with a pipe in the bottom of an open cistern *aa*. Into this pipe *b* water,* under high pressure, of which there was a convenient supply, could be admitted gradually by a cock *c*. The apparatus expanded upwards from the pipe to its mouth *d*. In one apparatus used, this mouth had an external diameter of 3·35 inches, in the other it had an external diameter of only 2·25 inches. Over the mouth of the apparatus the membranes experimented on were placed, and tied on by a waxed hempen cord, around a broad rim *ee*, immediately beneath the mouth. That the apparatus

* Water is preferable in these experiments to air, because, when it is employed, there is less violent action at the bursting of the membrane.

acted in a fair and satisfactory manner was evident, from the observation that, in almost all the trials, the membrane tested did not burst where it touched the instrument, but in an arc of a circle crossing over the bulged out membrane; or, rarely, in a starlike manner. Connected, by a hollow arm, with the apparatus was a vertical tube *g*, with scale *f* of inches and tenths of inches. This tube contained a long column of air, confined in it by a short column of mercury. The rise of the column of mercury compressing the air in the tube indicated the degree of pressure applied to the internal surface of the membrane fixed over the mouth of the conical vessel *d*. Besides my own supply, I was kindly provided with fresh membranes by Dr LINTON and Mr VACHER.

The following table gives, in a categorical form, a narration of each of 100 experiments, as well as the chief calculations founded upon the data obtained from them. The first column gives the number of the trial. The second column gives the number of the set of membranes tested; and it will be seen that generally several experiments were made with the same membranes. The third gives the length of continuance of labour till the time when the membranes were ruptured. The fourth column gives the duration of the first stage of labour. The fifth gives the duration of the second stage of labour. The sixth column contains the state of the os uteri at the time of the rupture of the membranes. The seventh states the stage of labour in which the bag of waters was broken. The eighth, ninth, and tenth columns show how many of the three membranes were tested simultaneously. The eleventh, twelfth, and thirteenth columns show what membranes gave way in each experiment. The fourteenth column states the radius of the circular mouth of the apparatus to which the membranes were tied. The fifteenth gives the barometric pressure at the time of each trial; and it will be observed that the pressure occasionally required a correction which demands explanation. The column of mercury in the apparatus was generally very short, and no correction for its weight was required, the experiments not pretending to an extreme nicety; but occasionally (in the cases noted in the column of remarks) the column of mercury was too long to be neglected, and a correction was made for its length. The sixteenth column gives the length of the column of air enclosed in the vertical tube above the mercury. The seventeenth gives the contraction of this column of air, by the pressure of water which burst the membranes, acting on the short column of mercury. The eighteenth column gives the height of the membrane as it bulged above the mouth of the apparatus, expanded by the water pressure. The nineteenth gives the effective pressure of the water, at the moment of bursting of the membrane, in inches of mercury. The twentieth gives the diameter of the sphere, of which the membrane when bursting approximately formed a portion. The twenty-first column gives the pressure per square inch of the membrane at the time of the bursting of the membrane, or at the time of the experiment's failing from some cause, such as

TABLE OF EXPERIMENTS.

1. No. of Experiment.	2. No. of Case.	3. Length of Labour till Rupture of Membrane.		4. Length of First Stage of Labour.		5. Length of Second Stage of Labour.		6. State of *Os uteri* at time of Rupture.	7. Stage in which Rupture occurred.	8. Membranes Tested. Amnion.	9. Chorion.	10. Decidua.	11. Membranes Burst. Amnion.	12. Chorion.	13. Decidua.	14. Radius of Apparatus.	15. Barometric Pressure.
																a	*b*
		h.	*m.*	*h.*	*m.*	*h.*	*m.*	*inch.*									
1	1									×	×	×	×	×	×	1·675	29·5
2	„									×	×	×	×	×	×	„	„
3	„									×	×		×	×		„	„
4	„									×			×			„	„
5	2									×	×	×			×	„	„
6	„									×	×					„	„
7	„									×	×		×	×		„	„
8	„									×						„	„
9	„									×	×	×			×	„	„
10	„									×	×					„	„
11	„									×	×		×	×		„	„
12	„									×						„	„
13	3									×	×	×	×	×	×	„	„
14	„									×	×	×			×	„	„
15	„									×	×					„	„
16	„									×						„	„
17	„									×						„	„
18	„									×						„	„
19	4									×	×	×	×	×	×	„	„
20	„									×	×	×			×	„	„
21	4									×	×			×		„	„
22	„									×						„	„
23	„										×			×		„	„
24	5	28	35	28	30	0	25		2nd	×	×		×	×		„	„
25	„	„		„		„			„	×	×			× ?		„	„
26	„	„		„		„			„	×			×			„	„
27	„	„		„		„			„	×	×		×	×		„	„
28	„	„		„		„			„	×			×			„	„
29	6	6	0	4	55	1	10		2nd	×	×			×		„	„
30	„	„		„		„			„	×			×			„	„
31	„	„		„		„			„	×			×			„	„
32	7	6	0						2nd		×	×		×	×	„	„
33	„	„							„		×	×		×	×	„	„
34	8	5	20	6	25	3	15	1½	1st		×	×		×	×	„	„
35	„	„		„		„		„	„	×			×			„	„
36	„	„		„		„		„	„	×			×			„	„
37	9	7	10	7	0	1	30		2nd	×			×			„	„
38	10	6	15	6	5	0	20		2nd	×	×	×	×	×	×	„	29·6
39	11	6	15	4	10	7	15		2nd		×	×		×	×	„	„
40	„	„		„		„			„		×	×		×	×	„	„
41	„	„		„		„			„		×	×		×	×	„	„
42	12	21	0	24	0	A few minutes		2½	1st	×			×			„	29·45
43	„	„		„		„		„	„	×			×			„	„

WITH THE DEDUCTIONS THEREFROM.

16.	17.	18.	19.	20.	21.	22.	23.	
Length of Column of Air.	Contraction of Column of Air.	Bulge of Membrane at time of Bursting.	Effective Pressure at Bursting, in inches of Mercury.	Diameter of Sphere of Membrane at Bursting.	Pressure on Square Inch.	Pressure on a Circular Surface of 2·25 inches radius.	Tensile Strength of Membrane.	REMARKS.
l	λ	h	$\frac{b\lambda}{l-\lambda}$.	$h+\frac{a^2}{h}$.	p $\cdot49\frac{b\lambda}{l+\lambda}$.	$1\cdot78\frac{b\lambda}{l-\lambda}\cdot\left(h+\frac{a^2}{h}\right)$.	T $\cdot123\frac{b\lambda}{l-\lambda}\cdot\left(h+\frac{a^2}{h}\right)$.	
19·75	2·		3·32		1 63			
,,	2·		3·32		1·63			
,,	1·		1·57		·77			
,,	3·25	1·5	·58	3·37	2·85	33·87	2·41	
,,	·5		·766		·37			Membrane taken close to placenta.
,,	1·5		2·42		1·19			Membrane slipped out.
,,	·75		1·16		·57			
,,	1·25		1·99		·97			Membrane slipped out.
,,	1·		1·57		·77			
,,	1·5		2·42		1·19			Membrane slipped out.
,,	1·25		1·99		·97			
,,	2·		3·32		1·63			Membrane slipped out.
20·	1·25		1·96		·96			Membrane taken close to placenta.
,,	·75		1.15		·56			
,,	1·2		1.88		·92			
,,	·6		·912		·45			
,,	·4		·602		·29			
,,	·6		·912		·45			
,,	1·1		1·72		·84			
,,	·5		·756		·37			
,,	·5		·756		·37			
,,	·8		1·23		·60			A considerable leak in the membrane.
,,	·2		·298		1·46			
16·5	2·5	·75	4·98	4·49	2·46	37·58	2 75	Barometer corrected by 1·6 inch.
,,	2·	·75	3·78	4·49	1·85	29·36	2·09	Barometer corrected by 1·6 inch.
,,	2·25	1·	4 40	3·80	2·16	28·96	2·06	Barometer corrected by 1·6 inch.
,,	2·25	·9	4·40	4·02	2·16	30·76	2·17	Barometer corrected by 1·6 inch. A leak in the membrane.
16·25	2 5	1·	5·07	3·80	2·48	33·37	2·37	Barometer corrected by 1·6 inch.
,,	1·5	·75	2·83	4·49	1·39	15·06	1·07	Barometer corrected by 1·6 inch.
,,	1·	·75	1·83	4·49	·90	14·21	1·01	Barometer corrected by 1·6 inch.
16·5	2·25	·75	4·40	4·49	2·16	31·18	2·43	Barometer corrected by 1·6 inch.
16·4	·6	·75	1·06	4·49	·52	8·23	·58	Barometer corrected by 1·6 inch. Birth 30 min. after rupture
16·3	5	1·	·882	3·80	·43	5·81	·41	Barometer corrected by 1·6 inch.
16·5	·4	·75	·693	4·49	·34	5·38	·38	Barometer corrected by 1·6 inch.
,,	·9	1	1·61	3·80	·79	10·60	·75	Barometer corrected by 1·6 inch.
,,	·9	1·25	1·61	3·49	·79	9·73	·69	Barometer corrected by 1·6 inch.
16·2	·6	1·	1·07	3·80	·52	7·04	·50	Barometer corrected by 1·6 inch.
,,	·3	·75	·526	4·49	·26	4·08	29	Barometer corrected by 1·6 inch.
20·	·75	·75	1·15	4·49	·56	8·93	·63	
,,	·75	·75	1·15	4·49	·56	8·93	·63	
,,	·4	·25	·604	11·47	·30	11·99	·85	
20·5	1·8	1·	2·83	3·80	1·39	18·63	1·32	
,,	·9	·75	1·35	4·49	·66	10·48	·74	

TABLE OF EXPERIMENTS,

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.
44	"	"	"	"	"	"	×			×			"	"
45	13	13 0	12 45	0 45		2nd		×	×			×	"	"
46	"	"	"	"		"		×			×		"	"
47	14	11 0	10 15	0 30?		2nd	×			×			"	"
48	"	"	"	"		"	×						"	"
49	"	"	"	"		"	×			×			"	"
50	"	"	"	"		"	×			×			"	"
51	"	"	"	"		"	×						"	"
52	"	"	"	"		"	×						"	"
53	15			7 30		1st	×	×					"	28·8
54	"			"		"	×	×			×		"	"
55	"			"		"	×						"	"
56	"			"		"	×			×			"	"
57	16	6 30				2nd	×			×			"	29·0
58	"	"				"	×			×			"	"
59	17	3 30	9 30	1 0	½	1st	×			×			"	"
60	"	"	"	"	"	"		×	×		×	×	"	"
61	18						×						"	"
62	"						×			×			"	"
63	"						×			×			"	"
64	19		6 0	1 0		2nd	×			×			"	29·2
65	"		"	"		"	×			×			"	"
66	"		"	"		"	×			×			1·125	"
67	"		"	"		"	×			×			"	"
68	"		"	"		"	×						"	"
69	"		"	"		"	×			×			"	"
70	"		"	"		"	×			×			"	"
71	"		"	"		"	×			×			"	"
72	"		"	"		"		×	×		×	×	"	"
73	20	9 15	8 20	0 55		3rd		×	×		×	×	"	"
74	21		9 0	3 0		"	×			×			"	"
75	"		"	"		"	×			×			"	"
76	22	1 30	2 0	1 0		2nd	×			×			"	29·8
77	"	"	"	"		"	×			×			"	"
78	"	"	"	"		"	×			×			"	"
79	"	"	"	"		"	×			×			"	"
80	23		Long	1 0			×			×			"	"
81	"		"	"			×			×			"	"
82	"		"	"			×			×			"	"
83	"		"	"			×			×			1·675	"
84	24	32 30	30 0	4 0		2nd	×			×			1·125	"
85	"	"	"	"		"	×			×			"	"
86	"	"	"	"		"	×			×			"	"
87	"	"	"	"		"	×			×			"	"
88	"	"	"	"		"	×			×			1·675	"
89	25						×	×		×	×		1·125	29·84
90	"						×	×		×			"	"
91	"							×			×		"	"
92	"						×			×			"	"
93	"						×			×			1·675	"
94	"						×			×			"	"
95	"						×			×			"	"
96	"						×						"	"
97	"						×			×			"	"
98	26	10 10	8 30	2 0		2nd	×			×			1·125	"
99	"	"	"	"		"	×			×			"	"
100	"	"	"	"		"	×			×			"	"

WITH THE DEDUCTIONS THEREFROM—*continued*.

16.	17.	18.	19.	20.	21.	22.	23.	
"	1·3	·8	1·99	4·31	·97	14·83	1·05	
"	·5	1·	·736	3·80	·36	4·84	·35	
"	·45		·661		·32			
"	·7	1·	1·04	3·80	·56	6·85	·49	
"	·8	1·5	1·19	3·37	·58	6·94	·49	Membrane slipped out.
"	2·	2·	3·18	3·40	1·56	18 72	1·33	
20·75	1·5	1·5	3·29	3·37	1·12	13·35	·95	
20·5	1·5	1·5	2·32	3·37	1·14	13·52	·96	Membrane slipped out.
"	2·	2·	3·18	3·40	1·56	18·72	1·33	Membrane slipped out.
21·	1·5	1·	2·21	3·80	1·08	14·55	1 03	Membrane found to have been injured.
"	1·25	1·	1·82	3·80	·89	11·98	·85	
"	2·25	2·	3·45	3·40	1·69	20·31	1·44	
"	1·25	1·	1·82	3·80	·89	11·98	·85	
"	2·7	1·1	4·28	3·65	2·10	27·03	1·92	Child born 30 minutes after rupture.
"	2·5	1·	3·92	3·80	1·92	25·80	1 83	
21·2	·5	·5	·700	6·11	·34	7·40	·53	
"	·3	·5	·416	6·11	2·04	4·40	31	
20.4	1·1	·8	1·65	4·31	·81	12·29	·87	Membrane slipped out.
"	1·25	1·	1·89	3·80	·93	12·44	·88	
20 5	2·3	1·5	3·66	3·37	1·79	21·34	1·52	
19·4	1·5	1·	2·45	3·80	1·20	16·13	1·15	
"	1·6	1·5	2·62	3·37	1·28	15·27	1·09	
22 3	2·	·75	2 87	2 44	1·41	12·10	·86	
"	1·7	·5	2·41	3·03	1·18	12·64	·90	An ill-conducted experiment.
"	2·3	1·1	3·36	2·25	1·65	13·08	·93	
"	1·9	·75	2·72	2·44	1·33	15·68	1·11	
"	2 5	·75	3·69	2·44	1·81	15·56	1·11	Burst by air contained between the membrane and water.
"	1·7	·55	2·41	2·85	1·18	11·88	·84	Burst by air contained between the membrane and water.
"	·9	·5	1·23	3·03	·60	6·45	·46	
"	1·2	·6	1·66	2·71	·81	7·78	·51	Membranes not ruptured till after birth of head.
22·2	1·2	·6	1·67	2·71	·82	7 83	·56	Membranes ruptured with first pain of labour.
"	1·7	·8	2·42	2·38	1·19	9 97	·71	
22·	2·	·5	2·98	3·03	1·46	15·62	1·11	A large leak in the membrane.
"	2·4	·65	3·65	2·60	1·79	16·40	1·17	A small leak in the membrane.
"	3·2	·75	5·07	2·44	2·48	21·37	1 52	
"	2·85	·6	4·43	2·71	2·17	20·76	1·48	
"	2 3	·7	3·48	2·51	1·70	15·10	1·07	
20·	1·8	·6	2·91	2·71	1·43	13·64	·97	
"	2·1	·5	3·49	3·03	1·71	18·30	1·31	A small leak in the membrane.
19·3	1·5	1·05	2·51	3 72	1·23	16·16	1·15	
19·8	1·	6	1·58	2 71	77	7·40	·53	Membranes ruptured in labour by attendant.
19·5	1·4	·7	2·30	2·51	1·13	9·98	·71	Membranes ruptured in labour by attendant.
"	1·7	·75	2·84	2·44	1 39	11·97	·85	Membranes ruptured in labour by attendant.
"	·9	·55	1·44	2·85	·71	7·10	·50	Membranes ruptured in labour by attendant.
19·3	1·4	·85	2·33	4 15	1·14	16·73	1·19	Membranes ruptured in labour by attendant.
22·	3·4	·65	5·36	2·60	2·63	24·08	1·71	Barometer corrected by ·5 inch.
"	3·6	·8	5·74	2·38	2·81	23·65	1·68	Barometer corrected by ·5 inch.
"	1·8	·9	2·61	2·31	1·28	10·41	·74	Barometer corrected by ·5 inch.
"	3·9	·75	6·32	2·44	3·10	26·64	1·89	Barometer corrected by ·5 inch.
19·2	1·25	·8	2·08	4·31	1·02	15·50	1·10	A leak in the membrane.
19·	1·3	·8	2·19	4·31	1 07	16·32	1·16	
"	1·45	·85	2·46	4·15	1·20	17·66	1·26	
"	2·	1·5	3·51	3·37	1·72	20·46	1·45	Membrane slipped out.
"	1·7	1·	2·93	3·80	1·44	19·29	1·37	Same piece as used in last experiment.
22·	1·5	·6	2 15	2·71	1·05	10·07	·72	Barometer corrected by ·5 inch.
"	1·6	·7	2·30	2·51	1·13	9·98	·71	Barometer corrected by ·5 inch.
"	1·8	·6	2·61	2·71	1·28	12 23	·87	Barometer corrected by ·5 inch.

the slipping of the membrane. The twenty-second column gives the pressure on a circular surface of 2·25 inches radius, or equal to the assumed dimensions of the lumen of the passage through which the child is expelled. The twenty-third column gives the tensile strength of the membrane, or, in other words, the weight which a band of it, an inch broad, would bear without giving way.

Professor TAIT has supplied the following formulæ from which the columns of the tables are computed:—

Let b be the height of the barometer, corrected for the short column of mercury in the gauge;

l the length of the air-column before pressure is applied;

λ the contraction of the column, when the membrane bursts.

Then, since the weight of a cubic inch of mercury, at ordinary temperatures, is about 0·49 lbs., we have, for the difference of pressures on opposite sides of the membrane when it bursts, the expression

$$p = 0{\cdot}49\, b\left(\frac{l}{l-\lambda} - 1\right) = 0{\cdot}49\,\frac{b\lambda}{l-\lambda}, \qquad (1)$$

in pounds per square inch. No sensible correction is required for the length of the water-column, when the mercury in the gauge and the membrane were not exactly at the same level.

If T be the force in pounds weight which will just snap a band of the membrane an inch broad, ρ the radius of curvature when the membrane bursts, we have, by a known theorem, the membrane being supposed to form approximately a portion of a sphere,

$$\frac{2\mathrm{T}}{\rho} = p \qquad (2).$$

To find ρ, we remark that the external semidiameter of the apparatus a is the radius of the base of a spherical segment, whose height h is measured; and geometry gives at once the equation

$$2\rho = h + \frac{a^2}{h} \qquad (3).$$

Hence, the tensile strength of the membrane is

$$\mathrm{T} = 0{\cdot}123\,\frac{b\lambda}{l-\lambda}\left(h + \frac{a^2}{h}\right) \qquad (4).$$

If we assume that the membrane is usually burst, by natural processes, when a portion of it forms a hemisphere of 2·25 inches radius, the requisite pressure in pounds per square inch will be, by (2) and (4)

$$\frac{0{\cdot}245}{2{\cdot}25}\,\frac{b\lambda}{l-\lambda}\left(h + \frac{a^2}{h}\right) \qquad (5).$$

and the effective pressure, on a circular surface of 2·25 inches radius, will then be

$$\pi\,(2{\cdot}25)^2\,\frac{0{\cdot}245}{2{\cdot}25}\,\frac{b\lambda}{l-\lambda}\left(h + \frac{a^2}{h}\right) = 1{\cdot}73\,\frac{b\lambda}{l-\lambda}\left(h + \frac{a^2}{h}\right) \qquad (6).$$

In making such experiments, a small given error in the estimate of the depth of the approximately spherical segment will be of least consequence, when the membrane bursts in a nearly hemispherical form, for by (3)

$$2\delta\rho = \delta h \left(1 - \frac{a^2}{h^2}\right)$$

and the error in the estimated radius vanishes, if $h = a$. Hence, also, the assumption that, in nature, the rupture takes place when the protruded portion of the membrane is hemispherical, gives a *minimum* value of the whole extruding force.

For the purposes of this paper the greatest value of the Table lies in the twenty-second column, which gives the power of the labour at the time of the rupture of the membranes and evacuation of the liquor amnii, on the supposition that the lumen of the passage opened up was circular, and of 4½ inches in diameter, and that the bulge was hemispherical at bursting. The first striking observation to be made, is the great variation in the strength of the bag of membranes. The force required to rupture the weakest amnion showed that the power of the labour was at least 4·08 lbs.; that for the strongest, a power of 37·58 lbs.; and the average power indicated by the experiments on the amnion, was 16·73 lbs. The average tensile strength was 1·19 lbs. Next, it is to be remarked, that in the cases whose membranes were tried, the power of labour almost certainly exceeded the power required to burst the bag, for it is not probable that a particularly weak small portion, unlike the rest of the membrane, was ruptured in the labours.

In cases 5, 6, 10, 14, 16, 26, the labour did not last above half an hour after the rupture of the membranes; and the greatest power indicated experimentally by rupturing the membranes was in each case respectively 37·58 lbs., 31·18, 4·08, 18·72, 27·03, 12·23.

In case 22, it was particularly observed by me that, so far as I could judge,* the pain rupturing the bag was stronger than any that followed; it may therefore be supposed, that the power terminating labour little exceeded 21·37 lbs., the greatest power indicated by the experiments as rupturing the membranes.

It was only after conceiving the means above described for arriving at the conclusions of this paper, and after the plan of the apparatus had been made by Professor TAIT, that I fell in with an interesting and valuable paper by Dr J. POPPEL of Munich—"Ueber die Resistenz der Eihäute, ein Beitrag zur Mechanik

* The same contractile force of the uterus at different periods of labour, or, to be more exact, at different dimensions of the uterus, will produce greater internal pressure, and, consequently, greater expulsive force, as the uterus is smaller (*vide* equation (2), p. 646); and, the amount of muscular contraction being supposed to be the same, there may be no sign to the attendant or patient of the increase of power. On the other hand, the application of the same principle shows, that, when the curvature of the extruded portion of the membranes is greatest, the difficulty of rupturing them is also greatest. This occurs when the extruded portion is hemispherical; and it is on this supposition that the numbers in column 22 are calculated.

der Geburt," contained in the first part of the twenty-second volume of the "Monatsschrift für Geburtskunde" for 1863. This paper anticipates to a very great degree the plans and results here related. But it may be pointed out that Dr POPPEL has neglected to note some conditions of the experiment, which cannot be omitted without damaging materially the accuracy and value of the trials; especially, he has always supposed the membrane to burst when in a hemispherical form, which is certainly an error, and one whose tendency is always to make the strength of the membrane too little (*vide* equation (2)). He has attached some weight to the part of the amnion tested, considering that greater strength would accompany proximity to the placenta; but my experiments did not confirm this opinion.

Dr POPPEL's apparatus may be sufficiently, though not fully, described as follows:—The membrane to be tried he ingeniously fixed over one or other of two glass vessels, of the diameter of five centimetres or two inches, and of ten centimetres or four inches, respectively. The glass vessels were reagent glasses, from which the bottoms were taken off. The affixed membranes represented the bottoms of the reagent glasses. Into the corks of the glasses a long glass tube was passed. Through this tube mercury was poured into the bottle till it filled it, and mounted into the tube. Its height in the tube at the time of the bursting of the membranes was carefully noted, because from it was estimated the pressure that burst the membrane. In adding the mercury fitfully, Dr POPPEL erroneously supposed that he imitated the pains of labour, a point, it appears to me, of no importance; and besides, his idea was manifestly erroneous, for each succeeding pain is not an addition to a force previously in action—it may even be weaker than its predecessor. In every natural case it is an entirely new force, rising in strength from zero to its acme, and again gradually fading to zero. Dr POPPEL made allowance for the weight of mercury contained in the reagent glass, over and above what was in the vertical glass-tube; but he neglected the important element of the degree of bulging of the membrane or radius of its curvature at time of bursting, with a view to arriving at the diameter of the globe, of which it formed a section at the time of rupture. With this he connects also a statement, that the bulging of the membranes through the mouth of the womb rarely exceeds a hemispherical form, which, though perhaps nearly true, is misleading, if held to be true in regard to the class of cases of persistent membranes specially studied in this paper.

The average strength of the amnion found by POPPEL was, keeping an aperture of 2·25 inches in radius in view, 19·21 lbs.; in my experiments it was 16·73 lbs.

POPPEL experimented on the membranes in seven cases in which they burst "with the birth." The following table gives the strength of the membranes in these cases, according to POPPEL's method of calculating, and the same changed

into lbs., as well as increased proportionally from what appertains to a radius of 5 centimetres to what appertains to a radius of 2¼ inches, the dimensions used in our experiments:—

No.	Belastung bei 10 Centimeter Durchmesser Kilogramm.	Pressure for Diameter of 2·25 inches in lbs.
	Kilogrammes.	lbs.
1	9·876	27·232
3	2·346	6·469
12	2·134	5·884
13	7·608	20·979
22	4·709	12·985
23	9·461	26·088
28	7·001	19·305

This table gives us, in seven cases, a figure of strength nearly equalling the whole power of labour in these cases. If, in any of the cases, the membranes had persisted after the birth, then the figure in the last column would have certainly exceeded the whole propelling power of labour at any moment during the whole of the labour. Speaking of them, POPPEL remarks, that "if we reflect that the table expresses only the minimum of power for the easiest labours, the figures appear to be quite trustworthy, even though they exhibit great variations. It may therefore be assumed that in a very easy labour a power, varying from 4 to 19 lbs., presses the head through the pelvis." As Dr POPPEL gives the passage transmitting the head a diameter of 4 inches, and as I prefer regarding it as nearer 4½, so I, using meantime POPPEL's experiments and calculations, make the power exerted in an easy labour vary from about 6 lbs. to about 27 lbs., instead of from 4 to 19. I shall not meantime attempt to show whether POPPEL's assumed 4-inch diameter or my assumed 4½-inch diameter is the more likely to be nearest the truth, because it would lead me into a class of questions remote from the subject matter of this paper.

If we observe, that in POPPEL's table of experiments and in mine the power shown to be sufficient to terminate an easy labour was often far exceeded in the course of other labours, we may enunciate the almost certain conclusion that a great mass of easy, and not merely of the easiest, labours is terminated by a power little in excess of that required to rupture the bag of membranes. The strongest membrane found in the experiments indicated, by the pressure required to burst it, an extruding force of 37½ lbs. We may therefore, I think, safely venture to assert as a highly probable conclusion, that the great majority of labours are completed by a propelling force not exceeding 40 lbs.

If we regard the figure of 4 lbs. given by POPPEL as equal to the power exerted in the easiest labour he has observed, or the corresponding figure of 6 lbs. according to my calculations, and keep in mind that the average weight of the

adult fœtus exceeds either of these weights, we are led to the conclusion that in the easiest labours almost no resistance is encountered by the child; that it glides into the world propelled by the smallest force capable of doing so; that, with the mother in a favourable position, the weight of the child is enough to bring it into the world—a result which many clinical facts at least appear to confirm.

Having thus given POPPEL's and my own estimate of the force exerted in natural parturition of the easiest kind, I can at present offer nothing positive from which to calculate the strength of labour in the general run of cases. My belief is that in ordinary labours the power exerted is not in general much above the lower limit; but other accoucheurs may see reason to entertain different opinions.

The higher limit of the power exerted in natural parturition has been variously estimated. There is an easy and obvious method of arriving at it. Cases are frequently occurring in which labour is artificially terminated by forceps, in circumstances which leave no doubt that, under delay, they would have come with difficulty to a spontaneous conclusion. The power exerted by the forceps in such cases can be measured. Such measurements are not to be at once taken as the power of labour necessary to finish such cases; but when all of the various sources of error are considered and included, they are of much value. The chief of such sources of error are the neglect of the assistance that may be afforded to the operator by the natural expulsive efforts, and the including of such forces exerted by the forceps as may be unnecessary for carrying on the process; for example, prematurely applied force, or force applied so as to advance the birth too hastily, or force lost by being used in a wrong direction. For the making of observations of this kind by the forceps special instruments have been invented by KRISTELLER and JOULIN.

But forceps-cases do not afford the only evidence available as to the higher limit. Experiments can be made on the dead subject which can be very well relied upon, as reproducing correctly the difficulty encountered in the living, and the power required to overcome it. Such experiments have been made by JOULIN,* and when suitably arranged, give us the power exerted in cases which may be spontaneously terminated by the most powerful parturient efforts; and, it may be added, with great risk of the mother's life.

Speaking of these experiments, JOULIN makes the following remarks:†—"Spontaneous delivery has been sometimes observed in circumstances almost identical. It appears to me, therefore, possible to admit that the figure of 50 kilogrammes (about a hundredweight) of force represents very nearly the maxi-

* Mémoires de l'Académie Impériale de Médecine. Tome xxvii. p. 90, &c. See also his Mémoire sur l'emploi de la Force in Obstétrique. Archives générales de Médecine: numéros Février et Mars 1867.

† Traité complet d'Accouchements, p. 477.

mum of the contractile power of the uterus; for it is necessary to take into account the accessory contingent furnished by the abdominal muscles, which in these instances was awanting. But as this force has not a direct action, it is probable that its actual product scarcely rises above a few kilogrammes."

Having had extensive and varied experience in the use of the forceps in difficult labours, and having also made some rough experiments with the dynamometer, to ascertain the power I have applied by the instrument, I regard M. JOULIN'S estimate of a hundredweight as the maximum force of the parturient function as far too high. I do not deny that, in very rare cases, such a force may possibly be produced; but I am sure that it is nearer the truth to estimate the maximum expulsive power of labour (including, with the uterine contractions, the assistant expulsive efforts) as not exceeding 80 lbs.

At present, I can divine no method of arriving at an estimate of the expulsive power of ordinary labours, except the following; and I must guard myself from being supposed to recommend its use, in the meantime at least. A fine tube, filled with water and of resisting material, may be introduced into the small pool of liquor amnii which remains after the rupture of the membranes filling up the spaces otherwise vacant on the anterior aspect of the fœtus. This tube should be provided with an aperture at its uterine end; it should be curved, so that when introduced it may lie easily in the pelvis, occupying the least possible space, so that no unnecessary resistance be offered to the advance of the fœtus; its wall should taper to either side, a cross section of it having a long pointed fusiform outline, in order that its presence may not produce on either side of it a channel for the running off of the pool of liquor amnii; lastly, its external end should be in communication with a column of mercury in a vertical tube, enclosing a column of air under only ordinary barometrical pressure. During the pains the rise of the mercury in the tube may be measured, and calculations from these measurements might be made identical with those already given in an earlier part of this paper. By this means, if successfully applied, the force of any labour may be exactly known. And it is scarcely necessary even to suggest how immeasurably valuable to the accoucheur such an estimate would be, substituting, as it would, an experimentally accurate statement of awful importance for the vague notions at present relied on, even when the wisest and most experienced practitioner lends his counsel.

I have already expressed my opinion as to the great practical importance of the inquiry entered upon in this paper. Although it is, as yet, far from completed, there is enough demonstrated to enable Dr SLOP, if he have an opportunity, to cast ridicule on the father of Tristram Shandy, who, founding on the statements of LITHOPŒDUS SENONENSIS, asserts, that the force of a woman's efforts is, in strong labour-pains, equal, upon an average, to the weight of 470 lbs. avoirdupois, acting perpendicularly upon the head of the child !!

Fig. 1.

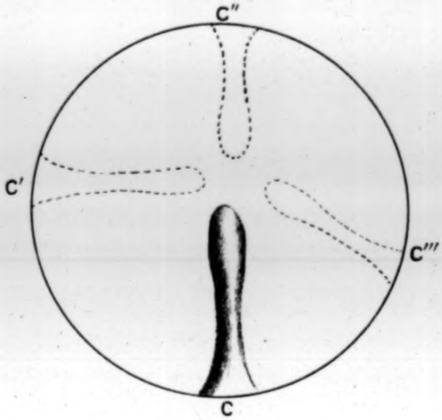


Fig. 2.

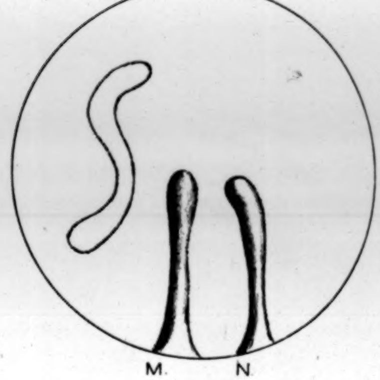


Fig. 3.

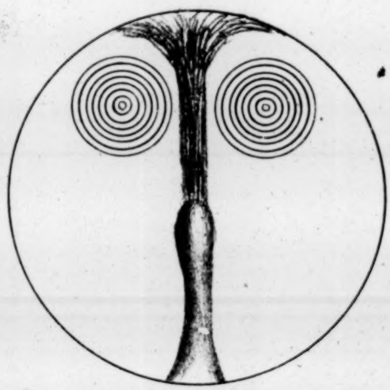


Fig. 4.

Fig. 5

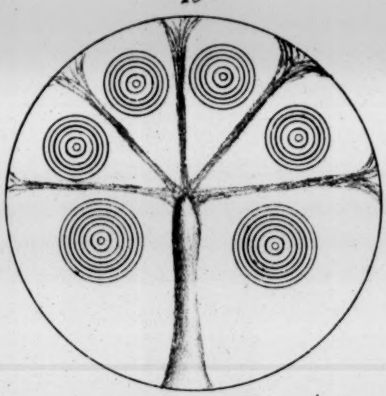
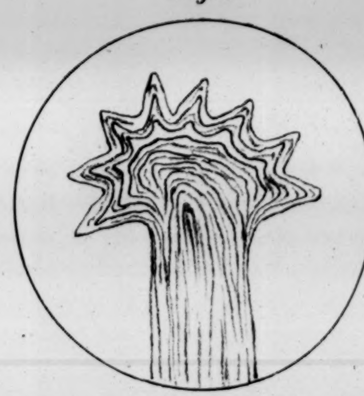


Fig. 6.



XLIII.—*On the Motions and Colours upon Films of Alcohol and Volatile Oils and other Fluids.* By Sir David Brewster, K.H., F.R.S. (Plate XXXIX.)

(Read 4th March 1867.)

In a paper "On the Phenomena of Thin Plates of Solid and Fluid Substances exposed to Polarised Light," published in the "Philosophical Transactions" for 1841,* I had occasion to notice certain motions and colours which I had observed upon films of some of the volatile oils; but as they were unconnected with the subject I was then investigating, I made no attempt to discover their nature and origin. Their apparent similarity, however, to the molecular movements and colours upon the soap-bubble, induced me to resume the subject, and to examine them as exhibited upon films of various evaporable liquids, stretched over apertures differing in size, form, and substance.

If we place a drop of alcohol upon an aperture C, C′, C″, C‴, Fig. 1, held horizontally, about the *fifth* of an inch or less in diameter, a concave lens will be formed upon it. As the alcohol evaporates, a very small plane film will appear in the centre, and will gradually increase in size till it fills nearly the whole aperture. If we hold the film in a vertical position, and examine it by transmitted light, we shall see a current of fluid, C, rising from the circumference of the film, moving rapidly from one part of the circumference to another, occasionally taking a horizontal position, and sometimes descending from the apex and sides of the film, as shown at C′, C″, C‴.

The current C is sometimes broad and flat, and separates into two currents, M, N, Fig. 2, which dance, as it were, opposite each other, assume the form of the letter S, and turn heels over head when they quit the circumference of the film.

These apparent currents generally throw out secondary currents, as in Figs. 3, 4, and 5, and the whole of them continue in rapid motion, exactly like a transparent insect struggling to escape.

When an excess of fluid is placed in the aperture held vertically, it occupies the lower part of it, and the film, no longer circular, appears in the upper part; but notwithstanding this change in its form and condition, the currents upon it present the same phenomena.

If the film is formed upon apertures of an irregular shape, it has, of course, the same shape, but the form and motions of the currents are not changed.

In some cases small currents issue from a part of the circumference of the films opposite the principal current; and in other cases small globules drop from the extremity of the principal one.

* Phil. Trans. 1841, p. 43.

The currents which are produced upon films of the alcohol of commerce appear also upon those of absolute alcohol; but though the films are smaller and less persistent, the currents are more active and varied in their movements. Similar currents are produced upon films of various solutions containing alcohol, of alcoholic solutions containing water and sugar, and of a large number of volatile and fixed oils, which, through the kindness of Dr PLAYFAIR and Dr CHRISTISON, I have been enabled to examine. They appear also, but with less activity, upon films of a solution of New Zealand gum in oil of laurel—the remarkable fluid by which the late Mr DELARUE produced the brilliantly coloured papers which were shown at the Great Exhibition of 1851.

In all these experiments the principal current and its different ramifications are perfectly colourless, and consequently exceed the thickness in NEWTON's scale at which the colours of thin plates make their appearance. In good and persistent films slight colours appear between the secondary currents; but these, as we shall presently see, are the complementary colours of those seen by reflection.

The various phenomena which I have described may be seen in a magnified form, by placing the films in a beam of divergent light, and they might be exhibited to an audience by means of the magic lantern.

If we now examine the surface of the alcoholic film by reflected light, we shall observe a series of phenomena of a very different kind. The principal current and its branches will be seen almost as distinctly as by transmitted light; but they are accompanied with, or rather they produce, systems of coloured rings of great beauty, shifting their place on the film, expanding and contracting quickly, and rapidly changing their form and their colours. Each pair of systems has on one or both sides a secondary current which stops or disturbs the rotatory motion, which would be communicated to two systems by the action of a single current.

When the film is first formed, especially if it is a very small one, there is only one system which is maintained by the colourless fluid issuing from the margin, sometimes in closely packed bands, of very high orders of colours. The lowest colour is always in the centre of the system, but the central tint is never lower than the white of the first order. When the tint occurs which NEWTON calls the beginning of black, the film always bursts.

This single system of rings is finely seen in films of very old balsam of *copaiba* which I obtained from Dr CHRISTISON's Museum. The film was wholly occupied by a circular system contracting and expanding quickly, changing its central tint, becoming elliptical, and even of a crescent form, when pushed onward by the thicker fluid from the margin of the film. These movements were kept up for more than an hour without any rotatory motion, and had not the film burst from an accident, they might have continued much longer.

After the single system of rings has appeared upon a film of alcohol, the film

gradually increases in breadth from evaporation, and the principal current throws out a secondary current, as in Fig. 3, which throws the colouring matter into two systems of rings that seem to move, the one from right to left, and the other from left to right, like those produced on the soap film by a current of air.

Another secondary current gives rise to three systems of rings, as in Fig. 4, and several such currents to several systems, as in Fig. 5, in which the motions are so rapid that it is difficult to follow them.

In this condition of the film the principal current becomes flat, and itself becomes divided, as it were, into narrow bands, and the various systems of rings unite into one singular system, rudely represented in Fig. 6; the tints beyond the salient points being at the margin of the film, the *white* of the first order gradually rising to the higher tints of NEWTON's scale. The contracting and expanding motions of this system become slower and slower, and the film generally bursts, scattering its fluid round the aperture upon which it was formed.

In some cases, which are very rare, the whole system of rings disappears with the principal current—the film becomes perfectly quiet and colourless, and, as in the soap film, the colouring matter, in the form of bands and tadpoles, comes upon it from its margin, till the black of the first order covers the film, and causes it to burst. This is a very instructive result.

Owing to the irregular movements of the secondary currents, it is difficult to observe the direction of the motion of the systems of rings in Figs. 3, 4, and 5; but they certainly revolve in opposite directions, like those produced by an artificial current in the soap film. When there is only one system of rings it has not a rotatory movement, because there is no decided current to put the colouring matter in motion; but when there is a principal current, with several secondary ones, as in Figs. 4 and 5, the secondary currents prevent, by their opposite actions, the intermediate rings from revolving.

The systems of rings which I have described are seen with various modifications, in about 70 or 80 volatile and fixed oils, and other liquids, which I have had an opportunity of examining. In most of them the motions of the currents and rings are very rapid. In some a film cannot be obtained, as in the ground nut-oil, the oil from the *Sesamum orientale*, and the purified oil of bitter almonds. In others the film, though it exhibits the play of colours, does not exceed the 50th or 60th of an inch in diameter. In *Elaine*, from olive-oil, and in the oil from the *Jatropha curcus*, a very small film is obtained, which instantly becomes black, and disappears. In Mr DELARUE's solution of New Zealand gum in oil of laurel, the motions of the rings are very languid, but the film is very persistent; and, what is interesting, I observed a number of black particles pass across the rings, and break their outlines, showing that the colouring matter floated upon the elastic film, as in the soap-bubble. The same effect was sometimes produced by blowing upon the film.

In the preceding experiments the films were obtained upon apertures in zinc, wood, and card-board, and on small rings of glass, platinum, iron, copper, and brass; but in the films upon all these substances the phenomena were the same. The films were more persistent when the apertures which they covered were made in plates of greater thickness, and some slight modifications of the phenomena arose from the same cause.

In reviewing the preceding experiments, it is impossible to resist the conclusion that the colours have the same origin as those on the soap-bubble. In both cases a colourless fluid issues from the circumference of the film, and spreads itself into rings and coloured bands, which are constantly changing their form and their colour. In some films the colouring matter seems to be occasionally recombined with the film, and the colours to reappear, and vary till the film bursts under the black of the first order. In the soap film the motion of the colouring matter is comparatively languid, but the bands and rings are, to a certain extent, under the influence of gravity, assuming a horizontal position during the rotation of a vertical film; but some other influence must be sought for, in order to explain the rapid and long-continued play of colours which is exhibited in films of alcohol and the volatile oils.

XLIV.—*On the Sophists of the Fifth Century*, B.C. By Professor Blackie.

(Read 18th March 1867.)

One of the most remarkable phenomena in our recent historical literature is a tendency to whitewash all characters which had previously presented a black appearance; to prefer the intellectual divination of a subtle modern professor to the plain testimony of a sober old chronicler; and generally to unsettle all things that we had in previous ages been taught to look on as settled. That this tendency, dating from the gigantic excavations of Niebuhr and Wolf, had its origin in an honest love of truth, and a searching scrutiny of evidence, cannot be doubted. That its results have in the main been beneficial is equally certain; but, on the other hand, it is not to be denied that it has sometimes run into the most wanton excesses, and that it has tainted some of the most notable historical productions of our age with a vice which will render it necessary for a future generation to repeat the work now done from a broader point of view, and with a juster criticism. Among the great works which have not escaped this prevalent contagion must be named the History of Greece, by George Grote. In this work, while the democratic institutions of Athens have been vindicated in the most masterly manner, and the political tone of the work may be regarded as, on the whole, sound, the author has in some prominent sections blotted his pages with the peculiarly German rage of substituting conjecture for fact, and overriding testimony by theory. And in doing this he has not only acted more like a German than an Englishman, but he has in some instances proceeded far beyond the bounds of negative criticism and bold assertion which the best German writers have observed. In no part of his work does this tendency, not only to overdo, but altogether to invert the natural order of things, appear more prominently than in his chapter on Socrates and the Sophists. In this part of his work, while he presents himself to the general reader as the chivalrous champion of injured innocence, the accurate weigher of historical evidence sees only another instance of the wonderful effect of a favourite theory in blinding a sensible man to the truth which radiates from the strongest testimony. To the reader of Mr Grote's chapter it must certainly seem as if Socrates had spent his life most stupidly, if not most basely, in fighting with a class of men, of which he himself was one, the best among many good, and that Protagoras was a far more sensible man, and, at bottom, a much more profound philosopher than Plato. The effect produced by this chapter of the history has been rather increased than diminished by the distinguished historian's comment on the Protagoras and other dialogues

in his recent work on PLATO. Here also we are regularly given to understand that PLATO was a much overrated man, and that the true objects of human admiration are rather the men whom it was the constant object of his philosophy to refute. This is even a bolder stroke of what, borrowing a phrase from mathematicians, I may call the *invertendo* style of criticism, than any with which the world has been favoured from the disintegrating school of LACHMAN, KÖCHLY, and other trans-Rhenane commentators on the Homeric poems. They, at least, while they annihilated the poet, left us the poem to admire. Here, the divine objects of old reverence are thrown away as idols, and the old recognised idols are set up as the true God.

The great authority of Mr GROTE in all matters of Greek history, and the wide circulation of his work, render it expedient that a public contradiction should be given to his errors from as many independent quarters as possible; and, though I am perfectly satisfied with what I find written on this subject by an excellent scholar, Mr COPE of "Cambridge, in the Cambridge Philological Journal," vol. i., as also by Professor ZELLER, in his *Philosophie der Griechen*, Tübingen, 1856; yet, as my own opinions have been formed altogether independently, and are based on a careful study of PLATO, extending through a series of years, I have thought that a succinct statement of the bearings of this important historical question would not prove unacceptable to the members of this Society. I proceed, therefore, to make a short statement of Mr GROTE's views of this matter, followed by an equally short statement of how, from my point of view, his arguments ought to be met.

Mr GROTE ushers in the statement of his views by this general declaration—"I know few characters in history who have been so hardly dealt with as the Sophists; they bear the penalty of their name in its modern sense;" and the modern sense of the word, according to the whole tenor of the learned gentleman's argument, is about as far removed from the original and genuine sense, as the English word *demon* is from the Homeric word δαίμων. To restore the proper meaning, as he conceives, to this sadly misunderstood word, the learned historian brings forward, according to my analysis, five arguments.

(1) It is plain from PLATO himself—in this case we must suppose an unwilling witness—that many of the Sophists were excellent and sensible men, and in every way capable of being the instructors of youth.

(2.) In fact, the Sophists were the great teachers of the age to which they belonged; and SOCRATES owed his position and his influence altogether to being one of them. The great exhibition of young democratic energy which had culminated at Marathon, was now riding onward triumphantly to another and a higher development. Of this period of transition between the youth and the manhood of the Athenian intellect the Sophists were the natural, the necessary representatives, and the worthy spokesmen.

(3.) PLATO was a man of peculiar idiosyncrasy, a great intellect confessedly, but a crotchety pedant in some matters, and a transcendental dreamer in others. His witness—at bottom the only serious testimony against the Sophists—(for of the great jester ARISTOPHANES in such matters we need take no account) is consequently of no value, and cannot, without the grossest injustice, be quoted against such sober, sensible, and practical thinkers as PROTAGORAS and GORGIAS.

(4.) The immoral teaching, attributed to the Sophists, and set forth by PLATO through the mouth of CALLICLES in the Gorgias and THRASYMACHUS in the Republic, must be a figment; for the whole history of the Athenian democracy shows that such doctrines would have been utterly revolting to them, and men professing such doctrines never would have been allowed the slightest influence in the education of their sons.

(5.) The Sophists, in fact, as a body, had no peculiar system of morals, either bad or good; as little had they any system of philosophic doctrine. They were a profession, not a sect.

(6.) The standing objection made to the Sophists by PLATO, in almost all his Dialogues, that they were a venal and mercantile crew, because they taught philosophy for a fee, need scarcely require refutation at our hands, living, as we do, in a country where the expediency of payment for all sorts of professional work is universally recognised. One does not, indeed, see how the Sophists could have performed their duties as general Hellenic teachers, travelling from land to land, had they not exacted a considerable fee, if it were only to pay their travelling expenses.

These propositions, it will be seen, have a polemical aspect, as indeed it is both the vice and the virtue of Mr GROTE's book generally, that he is everywhere writing down an old view of Hellenic matters, and writing up a new one. In order, therefore, fully to understand the drift of his statements, we must set distinctly before us the old doctrine about the Sophists which he affects to have overturned; and though this might be done by a large array of testimonies from many quarters, it will be sufficient for our present purpose to cite two of the best known authorities, BRUCKER and GILLIES, who may be looked on as the generally recognised exponents of the ante-Grotian doctrine with regard to the Sophists. In his "History of Philosophy," vol. i. p. 549, the erudite old Augsburg theologian says:—"*Erant tum temporis Athenis Sophistæ, magistri docendi, quales Leontinus Gorgias, Thrasymachus, Protagoras Abderites, Prodicus Ceius, Hippias Eleus aliique, qui in eo potissimum artem consistere arrogantibus verbis jactabant, quemadmodum caussa inferior, dicendo superior evadere posset; id quod, docente* CICERONE *sententiarum magis concinnitate argutoque et circumscripto verborum ambitu quam eorum pondere efficere tentabant. Hinc homines vani, ambitiosi, avari, quique soli sibi sapere videbantur, et omnium disciplinarum cognitionem sibi arrogabant, non tantum hanc in utramque partem de quavis re proposita invictis argu-*

mentis disputandi artem publice exercebant, sed et magnificis eam promissis nobilem juventutem brevi tempore se docturos pollicebantur. Quae eo ardentius ad hos nugatores deproperabat, quod ita se utilissimam rationem discere posse speraret, populum in suas partes trahendi, et ex civium ad quos loquendum erat, judicio, et calculo summam rerum ad se trahendi, vel etiam in potestate semel acquisita, flexo populi per istam eloquentiam obsequio, se confirmandi." And GILLIES, in his well-known "History of Greece," vol. ii. p. 133, says, in distinct antithesis to Mr GROTE, that the "appellation *Sophist*, in its modern sense, pretty faithfully expresses their character," and that "their morality supplied the springs from which EPICURUS watered his gardens, and their captious logic furnished the arguments by which PYRRHO laboured to justify his scepticism."

Now, in reference to these opposing views, my assertion is, that the old view, though not exhaustive of the whole truth of the matter, and not recognising certain modifications which tend to soften the harsher lines of the portrait, is on the whole the right view; while the new view, if containing an element of correction in some secondary points, is on the whole a false and misleading view, or rather a total misrepresentation and inversion of the facts of the case. The proof may be given, disposing of Mr GROTE's six arguments in their order, as follows:—

(1.) The general character of the Sophists, in their capacity of public teachers, is in no wise affected by the fact that there were great differences in their personal characters, and that some of them, like PROTAGORAS, were, as the world goes, most respectable and reputable men. The scribes and Pharisees in the Gospel history were respectable and reputable enough, no doubt, or had at least many most respectable and reputable men in their body; but not the less were their doctrines false and their teaching pernicious. So much only we may grant to the learned historian, that if any one ever said that there were no men of average respectability among the Sophists, such an assertion is altogether unwarranted, and is contrary to the plainest indications on the very surface of PLATO.

(2.) A similar admission may be made with regard to the historical significance of the Sophists generally, without, in the slightest degree, trenching on the ground occupied by PLATO. That the Sophists, like everything else in the world, had their good side, might have been assumed, if it could not have been proved; and it is equally certain, that when once a body of men like the Sophists, or the scribes and Pharisees, or the Romish priests, gets a bad name, the defects of character out of which that bad name arose are apt to occupy the whole of the canvass in historical tradition, while their virtues are altogether forgotten, or even denied. Hence arises a necessity for a sort of justification; a justification, however, which, while it may be allowed slightly to qualify, does not in any wise nullify the unfavourable character of the original verdict. A sort of plea in extenuation of this class of men was therefore, in the very nature of the case, to have been expected; and I am indebted to Professor ZELLER's admirable *Geschichte*

der Griechischen Philosophie for a reference to two of the earliest authorities, in which this reaction in favour of the Sophists appears. The one is Meiners, in his *Geschichte der Wissenschaften*, published at *Lemgo* in the year 1782, and the other that of Hegel, in his lectures on the history of philosophy delivered on various occasions soon after the commencement of the present century. Professor Meiners (vol. ii. pp. 172–599) says, "The Sophists deserve not merely to be despised and denounced, but in many views they claim respect and eulogy—a recognition which even their most violent opponents have not refused. They were the great public teachers and enlighteners of Greece; they were a necessary link in the chain of intellectual life in Greece." But while admitting this, the same author says a little further on, that "their morality was right in the teeth of the Socratic morality," and that, "on a review of the whole matter, we must agree with Xenophon, Plato, Isocrates, and those who followed them, that the Sophists did their country more harm than good, and that they corrupted more hearts than they enlightened heads." This representation deserves special notice as contrasted with Mr Grote's; for, while it fully admits the extenuating circumstance, it does not deny the general truth of the crime charged. Hegel places the palliative circumstance in a stronger light; indeed, he purposely brings it into the foreground, as being, in his phraseology, the one "positive and truly scientific side" of the matter. But by this he means, not that the faults with which the Sophists are generally charged did not really exist, but that whatever faults a faulty thing may possess, its virtues are the only element in it which has any value to a philosophic mind. From this point of view he says, that "the Sophists were the teachers of Greece, by whom intellectual culture (*Bildung*) was brought into existence. They came into the place of the poets and rhapsodists, who were originally the only teachers. Religion in Greece did not teach. Priests offered sacrifices, soothsayers divined the future, but instruction is something quite different." This is admirable; but with this the Berlin notional transcendentalist is far from shutting his eyes to the weak side of these teachers. He proceeds to represent them as practising a logic both superficial and unprincipled. He shows, also the peculiar danger which attached to such a logic when applied to practical purposes in an atmosphere of sensual polytheism. "In our European world," he writes, "intellectual culture appeared under the protection, so to speak, and on the foundation, of a spiritual religion. But when intellectual dexterity had to do only with a religion of the imagination, it readily shook itself loose from any central holding-point, or, at all events, particular subordinate points of view might easily be planted on the pedestal of an ultimate principle." And again, "A man of education and experience always knows how to set things in a good light for the momentary purpose. In the worst action something lies, which being singled out and skilfully presented, makes it defensible. A person must have gone a very short way in his intellectual education if he does not know how

to advance fair reasons to justify the worst actions. All the evil that has happened in the world since Adam has happened with the help of fair reasons." From these passages, which I think could not possibly be better expressed, we see how little the granting of Mr GROTE'S second argument has to do with the conclusion at which he so sweepingly arrives. The most comprehensive philosophical thinker of the most philosophic country in the world can see with the utmost distinctness that the Sophists were not all black, and yet that they dealt with the most important matters of human concernment in a loose and slippery fashion, which completely justified the attitude of uncompromising hostility constantly assumed towards them by both SOCRATES and PLATO.

(3.) Hitherto Mr GROTE'S arguments, so far as they present a mere plea in palliation of the Sophists, have appeared not only plausible, but in the highest degree reasonable; and, had he stopped at this point, there would have been no question at this moment before the learned world on this matter. But, unfortunately, the democratic historian here, by over-pleading his case, betrays the inherent weakness of his cause. He claims a verdict of acquittal for his clients, and can only do so, as we shall now see, by attempting to override an array of historical testimonies, such as, in the general case, would make any but a thorough-paced German ideamonger shrink back in dismay. The witnesses in this case are not few, and they are all on one side. Let us see how the learned historian disposes of them. In the first place, he throws PLATO and ARISTOPHANES, the greatest thinker, and the greatest humorist, of the age, *simpliciter*, out of court; and then, by either overlooking other testimonies, or referring them back to the twin authors of the original calumny, he tells the jury, with a gay confidence, that there is nothing more in the case. But there is a wholesale air about this procedure, which, with a sober-minded man, only acts as a warning to use caution. To commence with the two original framers of the indictment. No doubt ARISTOPHANES was a maker of jests, but he was no mere buffoon. He was a great thinker as well as a great humorist; and his comedies expressly deal with all the principal literary, philosophical, and political questions of the age. Such men are not apt to fling their humorous shafts at a mere imagination. On the contrary, their strength lies in the fact, that the phenomenon which they ridicule has a wide, popular recognition, and is everywhere felt to be a fact. A man of the calibre of ARISTOPHANES could not have written such a comedy as "The Clouds," against such a class of persons as the Sophists, had not such a class of persons existed, any more than the well-known scientific song of "The Origin of Species," attributed to a witty Scotch law-lord, could have existed without a DARWIN and a school of Darwins. The humorist's view of the case, indeed, is not necessarily the scientific view; but it may be, and often is, the true view, or, at all events, represents strongly one true aspect of the case. Otherwise, not only would the humour be pointless, but a great humorist certainly would not

meddle with the matter at all. Incidental errors, such as the confounding of Socrates with the mass of public teachers, of which he was one, do not affect the fundamental truth of the case. The "Clouds" is a play against the Sophists, not against Socrates.

But, however slight the value which a grave man may be inclined to give to the testimony of a great public humorist on a question of philosophy, if it stood alone, the case is completely altered the moment that his laughing testimony is confirmed by the serious witness of a professional thinker. The error which the greatest thinker and the greatest humorist of the age agree in condemning is not likely to have been an imagination. No doubt, in such a case, a great deal depends on the character of the philosopher; and Plato is not a name likely to forestall favour with a class of minds largely represented in this land, which rejoices to call itself pre-eminently practical, and shares in a more than Napoleonic hatred of all ideology. But let us distinguish. Plato undoubtedly had his crotchets: he was in some things a most unpractical man, and knew that he was so; unquestionably, also, his theory of ideas may often have been stated in exaggerated language, and with a paradoxical air, which were justly provocative of the opposition which, ever since Aristotle, it has encountered. But the testimony of the philosopher in reference to the Sophists is a thing much broader, and rooted much more deeply, than any of his crotchets about methodising the sexual instinct, or the possibility of his ideal polity. Here we have the fact that a great philosopher of all-commanding mind, the founder of a great and permanent school of thinking, who stood to his age in the same relation that Bacon does to ours, makes it the business of his life to write against, and represents his great master, Socrates, as having made it the business of his life, to speak against, a class of men who professed certain principles generally esteemed pernicious, but which, according to Mr Grote's view of the truth, were, in fact, most excellent and laudable. And this testimony, so given, was accepted by the universal voice of antiquity. It met, in fact, no decided contradiction till the epiphany of Mr Grote. Now, there is nothing altogether impossible in the supposition that Mr Grote may be right. It may sometimes be given to a Niebuhr, after a lapse of 2000 years, to reconstruct a history of Rome; but we are not to start with a prepossession in favour of such brilliant novelties. They are rather to be looked on with suspicion, and require strong backing. Plato, moreover, it must be borne in mind, with all his tendency to one-sided exaggerations, was by no means a narrow-minded, an ungenerous, much less a spiteful or ill-natured man. No man was more in the habit of looking at both sides of a question, and more unlikely to create a man of straw for an adversary. His treatment of Protagoras, Gorgias, and other Sophists, is what we would call gentlemanly in the highest degree, and gives the reader a sort of guarantee that what he alleges against the general body to which he belonged had some

good foundation. In weighing the testimony of PLATO and ARISTOPHANES also, with regard to such a class of men as the Sophists are alleged to have been, we must consider the presumptions and possibilities of the case. Is there anything strange or improbable in the statement, that in a talking town like Athens, full of all sorts of quick-witted and light-witted democratic people, there should have arisen, in an age of intellectual transition, a set of shallow thinkers, who cultivated the faculty of expression at the expense of the faculty of thinking, and exercised their understanding with a clever logical dexterity, rather than with the earnest search after truth? To myself it seems the most natural thing in the world to suppose the existence of such a class of men—a class of men, indeed, almost certain to exist at all times wherever there is a demand for them; and particularly dangerous, as HEGEL remarks, in a country where a sensuous religion exists, altogether divorced from any serious training, either of the intellect or the character.

Starting from these presumptions, I must confess I should be inclined to accept the portrait of the Sophists in every feature, and with its full colouring, as given by the god of the philosophers, and the king of the humorists, even if their testimony in this matter stood alone. But the plain and admitted fact here is, that neither the philosopher nor the humorist do stand alone; they are supported by the consenting voice of antiquity. The heritage of Greek opinion on this subject was transmitted to CICERO; and he says (Acad. II. 23), "*Sophistæ appellantur qui ostentationis aut quæstûs causâ philosophantur.*" Among the Greeks themselves, those whose testimony was of the highest value, and who lived nearest to the time, and who were most interested in the subject, set their seal in the strongest language to the witness of the great idealist. Who are the writers whom a wise judge would call into court, and hear with impartial eagerness in a trial of this kind? SOCRATES and XENOPHON, ISOCRATES and ARISTOTLE—any one of these would be sufficient, in my judgment, to nail down, for an absolute certainty, whatever PLATO and ARISTOPHANES might have previously combined to testify as a prominent fact in the history of Greek intellectual life. Of these four, though the most remote in point of time, ARISTOTLE is the most weighty; and this not only on account of the accurate, inductive, and encyclopædic character of his mind, but specially on account of his known propensity to contradict everything that PLATO says, when it comes in his way. None of the products of that peculiarly Platonic idiosyncrasy, which Mr GROTE brings forward so prominently, does the Stagyrite show the slightest desire to spare. Spartan women and Platonic ideas are two matters, in discussing which he almost seems to lose for a moment the imperturbable judicial coolness of his intellect. But the Sophists he describes in exactly the same language as PLATO, and in language which forms a sufficient justification for the peculiar use of the name in modern times. In Soph. El. I. 6, he says, "Εστι γὰρ ἡ σοφιστικὴ

φαινομένη σοφία οὖσα δὲ μὴ καὶ ὁ σοφιστὴς χρηματιστὴς ἀπὸ φαινομένης σοφίας αλλ' οὐκ ὅυσης."

The evidence of SOCRATES and XENOPHON need not be specified here in detail. They will be found below in a note, and have been admirably handled by Mr COPE in the Essay to which I previously alluded.* Only to the witness of ISOCRATES I call particular attention, as that of a man who was by the general bent of his mind not at all inclined to sympathise with any transcendental notions of high-strung intellectualists like PLATO, and who as himself one of the most reputable of the class of Sophists to whom GORGIAS belonged, would naturally feel no inclination to bring a charge against any large section of the fraternity, which might serve to increase the natural odium that in not a few quarters had always attached to the name. His words are as follows:—

"Τίς γὰρ οὐκ ἂν μισήσειεν ἅμα καὶ καταφρονήσειε πρῶτον μὲν τῶν περὶ τὰς ἔριδας διατριβόντων, οἱ προσποιοῦνται μὲν τὴν ἀλήθειαν ζητεῖν, εὐθὺς δ' ἐν ἀρχῇ τῶν ἐπαγγελμάτων ψευδῆ λέγειν ἐπιχείρουσιν; οἶμαι γὰρ ἅπασιν εἶναι φανερὸν, ὅτι τὰ μέλλοντα προγιγνώσκειν οὐ τῆς ἡμετέρας φύσεώς ἐστιν, ἀλλὰ τοσοῦτον ἀπέχομεν ταύτης τῆς φρονήσεως, ὥσθ' Ὅμηρος ὁ μεγίστην ἐπὶ σοφίᾳ δόξαν εἰληφὼς καὶ τοὺς θεοὺς πεποίηκεν ἔστιν ὅτε βουλευομένους ὑπὲρ αὐτῶν, οὐ τὴν ἐκείνων γνώμην εἰδὼς ἀλλ' ἡμῖν ἐνδείξασθαι βουλόμενος, ὅτι τοῖς ἀνθρώποις ἓν τοῦτο τῶν ἀδυνάτων ἐστίν.

"Οὗτοι τοίνυν εἰς τοῦτο τόλμης ἐληλύθασιν, ὥστε πειρῶνται πείθειν τοὺς νεωτέρους, ὡς, ἢν αὐτοῖς πλησιάζωσιν, ἅ τε πρακτέον ἐστὶν εἴσονταὶ καὶ διὰ ταύτης τῆς ἐπιστήμης εὐδαίμονες γενήσονται. καὶ τηλικούτων ἀγαθῶν αὐτοὺς διδασκάλους καὶ κυρίους καταστήσαντες οὐκ αἰσχύνονται τρεῖς ἢ τέτταρας μνᾶς ὑπὲρ τούτων αἰτοῦντες. ἀλλ' εἰ μέν τι τῶν ἄλλων κτημάτων πολλοστοῦ μέρους τῆς ἀξίας ἐπώλουν, οὐκ ἂν ἠμφισβήτησαν, ὡς οὐκ εὖ φρονοῦντες τυγχάνουσι, σύμπασαν δὲ τὴν ἀρετὴν καὶ τὴν εὐδαιμονίαν οὕτως ὀλίγου τιμῶντες, ὡς νοῦν ἔχοντες διδάσκαλοι τῶν ἄλλων ἀξιοῦσι γίγνεσθαι. καὶ λέγουσι μὲν, ὡς οὐδὲν δέονται χρημάτων, ἀργυρίδιον καὶ χρυσίδιον τὸν πλοῦτον ἀποκαλοῦντες, μικροῦ δὲ κέρδους ὀρεγόμενοι μόνον οὐκ ἀθανάτους ὑπισχνοῦνται τοὺς συνόντας ποιήσειν."

(4.) With regard to the moral teaching of the Sophists, Mr GROTE is quite right when he says that such an unblushing assertion of the doctrine that might is right, as is propounded by CALLICLES in the Gorgias, however welcome to DIONYSIUS in his rocky hold at Syracuse, would have been anything but agreeable to the Athenian democracy. But it is not necessary for those who consider that the Sophists were bad, and sometimes very bad moral guides, to maintain that they went about everywhere advocating despotic principles. PROTAGORAS,

* The contrast between the doctrine of SOCRATES and that of the Sophists, in reference to the origin of moral distinctions, is shown distinctly in the discussion between the former and HIPPIAS, in Xen. Mem. iv. 4, 13; and in the same work, i. 2, 6, the well-known objection to receiving μισθὸς for teaching morality, is stated by SOCRATES exactly as in PLATO. XENOPHON's own opinion is expressed very strongly in the last chapter of the treatise *De Venatione*: "'Οι δε σοφισταὶ δ' ἐπὶ τῷ ἐξαπατᾶν λέγουσι, καὶ γράφουσιν ἐπὶ τῷ ἑαυτῶν κέρδει, καὶ οὐδένα οὐδὲν ὠφελοῦσι, κ. τ. λ."

PRODICUS, and GORGIAS, and the other members of this notable brotherhood, whatever weak points their philosophy might offer to a sharp logician, were men of the world, and not likely to commence their teaching by plucking the beard of their audience, whatever that might be. Neither is there the slightest reason to suppose that all of them, or the majority of them, held immoral opinions with the same grand consistency with which their spokesman proclaims them in the Gorgias. The received doctrine with regard to the sophistical ethics which the learned historian undertakes to refute, is simply this, that by referring our ideas of right altogether to institution and convention, and in nowise to nature and divine necessity, they sapped the foundations of all morality, and made a justification of every iniquity easy to those who chose to argue consistently on their principles. And that there were plenty of men in Athens only too ready to carry such a doctrine to its legitimate practical conclusion, the unprincipled character of many public men in Athens, from ALCIBIADES to ÆSCHINES, sufficiently testifies. The character of the Athenian δῆμος may be placed as high as Mr GROTE, according to a democratic ideal, finds himself warranted to plant it; but it was not the δῆμος properly so-called, that is, the middle and lower strata of the Athenian people, by whom the principles of the slippery sophistical ethics were principally imbibed. It was the sons of the rich men, the oligarchy, the δύνατοι, that had most leisure and most ability to frequent the lectures of such men as PROTAGORAS, and to pay their fees; and how grandly they profited by their instructions, the oligarchic conspiracy of the four hundred in the year 411 B.C., and the government of the thirty tyrants, told to all the world with a signature of blood, whose significance Mr GROTE would be the very last man to misinterpret.

(5.) Mr GROTE's fifth argument, that the Sophists were not a sect or body of men like the Stoics and the Platonists, holding any particular set of opinions, but only a profession, like our modern literary men, critics, and reviewers, may be disposed of in a single sentence. Nobody ever said that they were a sect, but a class of men following a particular profession, and who were distinguished generally by a certain common character and principles. Of this the French Encyclopædists, to whom the Sophists have been aptly compared, were a notable example.

(6.) The matter of the μισθός, or fee which the Sophists charged for their instructions, must not be looked at from a merely modern point of view. The Sophists were not, like our professors, public servants engaged to give a certain special training to young men, either on receipt of a salary from the public, or of single fees from individual students. They came forward voluntarily with broad general professions, to fit men for public life, by teaching both the art of public speaking and all that effective speaking implies. They professed to teach the wisdom of life, the art of getting on, and especially the art of governing men in popular assemblies. This, it is evident, is a very serious matter, and very different from the attitude that belongs to any modern teacher. What they professed

to do could not be done scientifically without discussing the principles of right and wrong, and teaching virtue, ἀρετὴ in fact as well as ῥητορική. This is the point so ably brought out in the Gorgias. Now, the receiving of a fee for a large profession of this kind is a very different thing from paying a price for a pair of boots to a shoemaker, or for so many lessons in grammar to a language master. The question might be raised on the very threshold—*Can virtue be taught?* the famous question, εἰ διδακτὸν ἡ ἀρετὴ, discussed in the Menon and the Protagoras; and the strongest arguments were at hand to prove that, if it was teachable at all, it certainly was not to be taught in the same way that dancing may be learned from a dancing master, or music from a music master. A man goes to a teacher of Sanscrit, for instance, gets so many hours' grammatical exposition, appropriates the cram, passes his examination, gets an Indian appointment, and reposes comfortably upon more than the value of his fee. Here there is a definite *quid* for a definite *quo*, in the most distinct and mercantile sense. But the moral teacher must go to work in a different fashion. He does not offer a marketable article, and therefore cannot expect or demand a market price. For a mere course of lectures on the virtues, with which the scholar is to be duly crammed, will not do the business; it may prove worse than useless. A moral teacher must commence with teaching the student to see his faults, to confess his errors, and to amend his way. No man comes forward with a guinea in his hand to get instruction of this kind. No man expects to be paid for giving good but disagreeable advice to a conceited coxcomb, or a pompous pretender. And, accordingly, in our Christian churches clergymen are paid, not the value of their sermons, but, like the Platonic φύλακες, they receive a general salary for their maintenance. A sermon has no market value. No man paid the Hebrew prophets for their patriotic denunciations. The Athenians paid SOCRATES for his life-long speaking of all truth, and exposing of all sham, with a dungeon and a cup of hemlock. I therefore think that SOCRATES was right in refusing to receive a fee for teaching virtue. Besides, there is an element of convention in this matter which must not be overlooked. No public man in this country is paid, or would receive payment, for serving his country as a member of Parliament; and if PROTAGORAS, or any other accomplished speaker, came forward in Athens professing to teach virtue for a fee, the public conscience was entitled to be offended by the novelty, and to make a strict cross-examination of the individual who made such pretentious professions. One thing is certain, that not only in Athens, but in modern England and everywhere, the public teacher who demands no fee for his services, and can be suspected of not the slightest admixture of mercenary motives, must always stand upon a moral vantage ground that the paid teacher cannot occupy. This is the secret, or part of the secret at least, of the great influence exercised by WHITFIELD and other zealous evangelists in the last century, who, flinging away the golden hopes of ecclesiastical preferment,

devoted themselves to field preaching and missionary work among the most abandoned classes, by whom an entirely moral service could be repaid only by a moral reward.

This paper may be most fitly concluded by an articulate statement of the heads of the sophistical doctrine, as I abstract them from the works of PLATO, supported by the general testimony of the ancients :—

I. General information and alert intelligence without a philosophical basis, or a scientific method of verification.

II. The art of public speaking, considered merely as a means of moving masses of ignorant men with a view to political advancement, but not necessarily connected either with pure motives, lofty purpose, or business habits.

III. The exercise of a dexterous logic, that aimed at the ingenious, the striking, and the plausible, rather than the true, the solid, and the judicious.

IV. A theory of metaphysics which, by confounding knowledge with sensation, and subordinating the general to the particular, made wisdom consist rather in the expert use of present opportunity, than in the moulding of materials according to an intellectual principle.

V. A theory of morals which, by basing right on convention, not on nature, deprived our sensuous and animal passions of the imperial control of reason, and substituted for the eternal instinct of justice in the human heart the arbitrary enactments of positive law, whose ultimate sanction is the intelligent selfishness of the individual.

XLV.—*On the Diurnal Variation of the Magnetic Declination at Trevandrum, near the Magnetic Equator, and in both Hemispheres.* By JOHN ALLAN BROUN, Esq., F.R.S., late Director of the Observatory of His Highness the Maharajah of Travancore, G.C.S.I., at Trevandrum. (Plates XL. to XLIV.)

(Read 29th April 1867).

Preceding Observations and Conclusions.

The first observations of the diurnal variation of magnetic declination, made near the equator, seem to have been those of Mr MACDONALD, who observed in 1794–95 at Fort Marlborough, Sumatra, 3° 46′ S., and at St Helena. Two conclusions seem to have been deduced from these observations—1*st*, That near the equator the range of the diurnal variation was much smaller than in Europe; 2*d*, That the needle moved in opposite directions south of the equator and in Europe.* This latter conclusion was made use of by M. ARAGO, in his report made in 1821, on the "Voyage de l'Uranie," as the base of a hypothesis that there must be a line betwixt the two hemispheres on which the magnetic needle moves neither east nor west—that is, remains stationary. M. DE FREYCINET's observations showed that this line was not the terrestrial equator, and M. ARAGO supposed it must be the magnetic equator.†

In 1825 M. ARAGO again alluded to his hypothesis, in his report on M. DUPERREY's observations. These seemed to show—1*st*, That the diurnal variation was not extinguished on either equator; 2*d*, That in south latitudes, but with north magnetic inclination, the needle moved as in Europe. M. ARAGO then suggested, "Peutêtre les changements de declinaison du soleil, qui en Europe occasionnent de si grandes variations dans l'amplitude des oscillations diurnes, amenent ils suivant les saisons sous les tropiques des mouvements de l'aiguille dirigés en sens inverse."‡

M. ARAGO recurred to the subject in 1835, in his "Instructions pour la Bonite," and he suggests some additional line, some curve of equal magnetic intensity, as the curve of no diurnal variation.§

* Phil. Trans. abridged, vol. xviii. pp. 29 and 355. The first of the conclusions is given by Mr MACDONALD. He appears to have found, that the north end of the needle moved east from 7 A.M. till 5 P.M. The observations having been made from June 1794 till March 1795, the result from the observations for the first four months (June to September) should have been nearly the inverse of that from the observations for the remaining months. There can be little doubt that the instrument employed was incapable of showing the variations with much accuracy.

† Œuvres de F. ARAGO. Instructions, Rapports, etc., p. 152. Voyage de l'Uranie.

‡ *Ibid.* p. 196. Voyage de la Coquille. § *Ibid.* pp. 25, 26.

In his examination of observations which seem to have been made with much zeal and care, by M. DE TESSAN, in 1837–38, in different latitudes, at different periods of the year, M. ARAGO says, with reference to the question, whether the hours of maxima and minima of declination are identical over all the earth: "Nous pouvons affirmer qu'il n'en est pas ainsi l'aiguille horizontale atteint les limites de ses excursions diurnes, à des heures différentes suivant les climats."* With this affirmation M. ARAGO destroyed the base of his hypothesis. M. DE TESSAN himself pointed out, "Que la transition de l'un de ces états (the movement in one hemisphere), à l'état opposé (the movement in the other hemisphere), à mesure qu'on passe d'un hémisphère à l'autre, peut se faire aussi par un déplacement graduel des heures critiques auxquelles la direction du mouvement change. Et la realité de ce dernier mode de transition qui *a priori*, est aussi possible que l'autre me paraît indiquée avec beaucoup de vraisemblance par la succession régulière des courbes de notre tableau graphique."†

Yet we find in August 1840, after a study of all the observations made in the different voyages terminating with that of the "Venus," that M. ARAGO thought we should still seek the line of no diurnal variation, "Une courbe le long de la quelle l'aiguille par exception conservera de jour et de nuit absolument la même direction."—"Une courbe qui deviendra aussi l'objet de bien des recherches de bien des voyages."‡ It will be seen in his posthumous work on "Terrestrial Magnetism," that he preserved this idea to the last.

A consideration of the observations examined by M. ARAGO, of the shortness of their duration at each station, and of the difficulties of obtaining accurate observations with an imperfect installation of instruments (signalised by M. DE TESSAN), will explain his varying conclusions, and show that the materials in his possession were insufficient to answer the questions as to the mode in which the diurnal variation of the magnetic needle changed during the year at any station within the tropics, or from one latitude to another, at the same period.

The first series of observations of sufficiently long duration within or near the tropics, and capable of showing what laws the magnetic needle really followed there, were made in the Observatories of Bombay, Madras, Trevandrum, Singapore, and St Helena. The first, second, and fourth of these observatories were established at the expense of the East India Company, the last by the British Government, and the third by His Highness the Rajah of Travancore.

The observations made at Trevandrum, in June to December 1841, under the direction of the late Mr CALDECOTT, F.R.S., were in the possession of the Royal Society of London early in 1842, with the projected diurnal curves of magnetic

* Instructions, Rapports, etc., p. 283. Voyage de la Venus. Report, dated 1840.

† Voyage autour du monde sur la Fregate la Venus. Physique, vol. v. par M. DE TESSAN, 1844, p. 417. See also p. 461.

‡ Œuvres, Rapports. etc., p. 288.

declination for each month. Mr Caldecott communicated to that Society the fact shown in his curves, that the law of variation was, to a great extent, inverted in the course of October 1841. Although the fact that the law of diurnal variation for December was nearly the inverse of that for June within the tropics, must have been perceived early by the Directors of other intertropical observatories, yet Mr Caldecott was, as far as I am aware, the first who brought the fact with its proof to the notice of a scientific body.*

As will appear hereafter, none of the intertropical observatories, excepting that of Trevandrum, was sufficiently near the magnetic equator to approximate to a solution of the question proposed by M. Arago; and Cape Comorin, about 40 miles S.S.E. of Trevandrum, was one of the stations pointed out by this philosopher for that end.

Position of the Trevandrum Observatory.

The observatory at Trevandrum is in latitude 8° 30′ 32″ N. (about 2°½ S. magnetic inclination, and 26′ E. magnetic declination in 1854), and $5^h\ 7^m\ 59^s$, east of Greenwich, on a height nearly 200 feet above the sea. The following results are deduced from observations made under my direction, during the twelve years 1853 to 1864.

Instruments used.

Two declinometers were employed after 1853; one by Grubb of Dublin, similar to that described in the introductions to the Makerstoun Observations—one by Adie of London, made according to my own plans. With the former, all the precautions were taken indicated for the Makerstoun declinometer; and the telescope could be compared with the south transit-mark of the observatory (4 miles distant). The second was under a glass-receiver, from which the air was pumped; the suspension apparatus was supported by a tripod of glass rods, so that very little metal was used in the construction of the instrument. A room was built *within* the observatory for this declinometer, having a solid planked ceiling covered like a terrace with plaster, several feet below the roof. The declinometer was read from without the closed room, the telescope pillar forming part of a

* Mr Caldecott gave the diurnal curve for the first half, and for the second half of October. In Notes to the Observations, dated 10th November 1841, which were forwarded to the Royal Society of London, in the same month, he deduced,—

From the 1st half of October, max. of E. declin. 7^h A.M.; min. $0^h\ 28^m$ P.M.
„ 2d „ min. „ 8½ A.M.; max. $4^h\ 28^m$ P.M.

He attributed the change of law at the time to the monsoon, which broke out during the month; and he caused the observations to be continued every ten minutes, in the month of November, in order to determine whether the change would continue. Mr Caldecott's remark was never published by the Royal Society of London, but it will be found among the MS. of Trevandrum Observations, in the archives of the Society, where I have myself seen it. It is necessary to add, that Mr Caldecott's observations, though affected by many errors (chiefly due to the imperfect construction of the magnetometer boxes), were sufficiently exact for the determination of this change of law.

well-founded wall. The daily variation of temperature within the closed room was never more than a few tenths of a degree Fahrenheit. The value of a scale division of GRUBB's declinometer was 15″·35, and of ADIE's, 16″ nearly; tenths were easily estimated, and as the vibrations in ADIE's instrument were generally very small, if any, the mean of three readings was supposed accurate to one second of arc. Both declinometers were observed hourly, and one was observed twice at each hour, one minute before, and one minute after the other.*

Diurnal Variations near the Magnetic Equator.

TABLE I.—*Hourly Variations of Magnetic Declination at Trevandrum, deduced from Observations during the twelve years,* 1853 *to* 1864.

Hour, Trevandrum Mean Time.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sep.	Oct.	Nov.	Dec.
H. M	′	′	′	′	′	′	′	′	′	′	′	′
12 28	1·34	1·10	0·47	1·18	1·64	1·54	1·42	1·77	1·66	0·73	1·44	1·54
13 28	1·27	1·07	0·47	1·27	1·74	1·67	1·58	1·89	1·74	0·72	1·40	1·47
14 28	1·15	1·00	0·38	1·26	1·78	1·77	1·66	1·96	1·77	0·67	1·27	1·34
15 28	0·97	0·86	0·26	1·15	1·77	1·80	1·74	2·02	1·81	0·57	1·08	1·15
16 28	0·73	0·69	0·13	1·04	1·78	1·83	1·82	2·14	1·85	0·45	0·85	0·92
17 28	0·50	0·51	0·07	1·13	2·07	2·10	2·13	2·54	2·20	0·41	0·55	0·70
18 28	0·20	0·19	0·00	1·56	2·85	2·95	2·97	3·54	3·19	0·57	0·08	0·26
19 28	0·00	0·00	0·06	1·67	2·90	3·06	3·06	3·64	3·31	0·59	0·00	0·00
20 28	0·64	0·34	0·27	1·40	2·33	2·44	2·35	2·73	2·51	0·57	0·43	0·47
21 28	1·37	0·86	0·53	1·11	1·45	1·56	1·46	1·62	1·57	0·43	0·82	0·93
22 28	1·46	1·19	0·73	0·77	0·65	0·78	0·73	0·67	0·79	0·12	0·95	1·15
23 28	1·41	1·11	0·67	0·26	0·16	0·20	0·23	0·12	0·21	0·00	1·10	1·29
0 28	1·69	1·25	0·53	0·00	0·00	0·00	0·00	0·00	0·00	0·28	1·66	1·78
1 28	2·06	1·48	0·51	0·18	0·42	0·28	0·18	0·40	0·44	0·72	2·14	2·25
2 28	1·98	1·47	0·69	0·47	0·94	0·67	0·61	0·98	1·20	1·07	2·12	2·33
3 28	1·86	1·42	0·79	0·78	1·35	1·05	1·11	1·63	1·85	1·27	1·93	2·32
4 28	1·80	1·37	0·63	1·01	1·64	1·30	1·40	2·08	2·09	1·12	1·75	2·18
5 28	1·57	1·24	0·42	1·02	1·65	1·33	1·35	2·05	1·80	0·73	1·49	1·84
6 28	1·54	1·19	0·44	1·01	1·45	1·15	1·21	1·79	1·70	0·74	1·57	1·82
7 28	1·65	1·26	0·43	0·77	1·15	0·89	0·92	1·50	1·54	0·71	1·54	1·81
8 28	1·57	1·23	0·40	0·76	1·11	0·88	0·84	1·43	1·50	0·67	1·45	1·71
9 28	1·49	1·15	0·37	0·85	1·24	1·00	0·95	1·45	1·47	0·60	1·38	1·61
10 28	1·42	1·15	0·36	0·98	1·38	1·17	1·09	1·52	1·50	0·65	1·39	1·54
11 28	1·39	1·11	0·46	1·09	1·54	1·37	1·26	1·64	1·60	0·72	1·46	1·55

The quantities in this table will be found projected, Plate XL., a consideration of which has led to the following conclusions:—

1*st*, The diurnal variation consists of one marked maximum and one marked minimum of *easterly* declination in each month of the year, and of one or more secondary maxima and minima.

* The details connected with the description of instruments and the precautions taken to prevent errors, will be found in the first volume of the Trevandrum Observations, which I hope will soon be published.

2d, The principal maximum occurs in the six months of April to September at about 7 A.M., and the principal minimum occurs about twenty minutes past noon in the same months.

3d, Nearly the inverse of this happens in the four months of November to February, inasmuch as the principal minimum occurs between $7^h\ 10^m$ and $7^h\ 20^m$ A.M., and the principal maximum occurs betwixt $1^h\ 45^m$ and 3^h P.M.

4th, When we examine the mode in which this inversal takes place in the mean curves, it appears to happen in two ways:—*1st,* In a direct transition during the month of March from a minimum to a maximum at 7 A.M.; and during the month of October from a maximum to a minimum at the same hour; in which transitions the mean movement must necessarily at one time become zero about the critical hour. *2d,* In an apparent *shift* of the time of the afternoon maximum.

5th, The principal minimum, it has been noticed (*2d*), occurs in the months of April to September within a few minutes of twenty minutes past *true* noon; this minimum never wholly disappears in the mean curves at Trevandrum. Owing to the change going on in the law of variation in the months of March and October, it happens about 30^m before noon in the latter month, when it is still the principal minimum, and about one hour after noon in the former, when it is a secondary minimum; it appears as a slight minimum, or as inflexion only in the curves, for the four months of November to February. This conservation of the noon minimum would seem to prove that there is no real inversion at this hour, and that the minimum near noon and the maximum after noon are two distinct and independent phenomena, the former becoming more and more marked from January to July.

6th, The conservation of a minimum near noon in the six months of October to March, has, as a consequence, the appearance of a secondary maximum, or an inflexion of this character betwixt 7 A.M. and noon—generally near 10 A.M.

7th, The principal maximum occurs betwixt $1^h\ 45^m$ and 3^h P.M. in the four months November to February; in March and October it happens near $3^h\ 30^m$ P.M.; and in the remaining six months of April to September, from $4^h\ 20^m$ to $5^h\ 30^m$ P.M.

8th, A quite secondary minimum occurs near 6 P.M. (or almost exactly at sunset) in the six months of October to March; and a more marked minimum appears in the curves for the six months from April to September between 8^h and $9^h\ 30^m$ P.M. A faint minimum can be traced in the curves for the remaining six months, October to March, at nearly the same hour; so that the minimum at sunset is a phenomenon independent of that near 9 P.M.

9th, The north end of the needle moves between 8 P.M. and 7 A.M. towards the east in the five months May to September; it moves towards the west between the same hours in the four months November to February; whereas in the months of March, April, and October, the change from one direction to another

produces a secondary minimum near 5 A.M., and this minimum has its representative in an inflexion near the same hour in the curves for May to November.

10*th*, It might be suggested that the minimum or inflexion near 5 A.M. in the six months of April to October is due to the same cause as the minimum in the remaining six months at 7 A.M., and that there is here also a shift of the critical hour.* In any case there cannot be a doubt of the absolute inversal of the mean movements at 7 A.M. during the months of March and October, a phenomenon which has been observed at no other station.

These conclusions as to maxima and minima are deduced from curves which represent the *mean* movement corresponding to the *middle* of each month. I have had to point out, on different occasions, that deductions from mean values may be wholly inaccurate, since it sometimes happens that the observations represent laws which are very different, and the mean may represent none of them. It will be desirable, then, in this instance, to endeavour to determine in what way the law of variation really changes from day to day, especially in the months when the change is marked. I shall, however, consider previously the range of movement of the declination needle.

TABLE II.—*Ranges of the Monthly Mean Movements in Table I., and Monthly Means of all the Daily Ranges during the twelve years,* 1853 *to* 1864.

Months.	Range of Mean A	Mean of Ranges B	Ratio $\frac{B}{A}$	$\frac{B}{1\cdot11}$
January,	2′·06	3′·09	1·50	2·78
February,	1·48	2·54	1·72	2·29
March,	0·79	2·07	2·62	1·86
April,	1·67	2·35	1·41	2·12
May,	2·90	3·23	1·11	2·90
June,	3·06	3·40	1·11	3·06
July,	3·06	3·45	1·13	3·06
August,	3·64	4·02	1·10	3·64
September,	3·31	3·68	1·11	3·31
October,	1·27	2·25	1·79	2·03
November,	2·14	2·70	1·26	2·43
December,	2·33	3·04	1·30	2·74

Ranges of the Mean Movements of the Declination Needle near the Magnetic Equator.

It is evident from the quantities A and from the curves, Plate XL., that there is a tendency to an extinction of the *mean* movement in March and October, and especially so in the former month. The range of the *mean* movement is a minimum in these two months, and it is a maximum in August and December.

The range is greater in August than in December. This fact, and the exist-

* This point will be examined afterwards when the lunar effect has been eliminated.

ence of an inflexion or slight minimum near noon in the months about December, seem to show that Trevandrum belongs more properly to the northern hemisphere (to which it belongs geographically), than to the southern hemisphere, to which it belongs magnetically.

In comparing the curve of February with that of April, and that of September with the curve of November, it would seem probable, that at some other epoch than the middle of the months of March and October, the amount of the range would be still less than that here shown. A consideration of monthly mean curves, corresponding to each week of the year, shows that this is the case; but as the date of minimum movement is not the same in each year, a combination of the means at the same date in different years will not show the smallest amount of the movement. I have, however, projected (Plate XL.) the mean curve corresponding to the middle of March 1864, which shows the mean movement limited to about one-third of a minute of arc (the range of the hourly observations is 0′·35).

We have thus a very near approximation to an extinction of movement in the mean curve for the month of March 1864; and if the mean curve represented the usual daily movement, we might say that we have here a case in which the needle remains stationary night and day. This is, however, not the fact; the result is to a considerable extent arithmetical, and is due, in part at least, to a combination of different movements.

In the third column (B) of Table II. I have given the means of *all* the *daily* ranges for each month. It will be perceived from these, that though the diurnal oscillation is still a minimum in March and October, yet it is a much more marked oscillation than appears in the mean curves. In the year 1856, for which the mean of the daily ranges is least, the mean of those for March is 1·51 and for October 1·76; these, though very small mean ranges, are still more marked than we might expect from the mean curves.

The fourth column of Table II. contains the ratio of the mean of the daily ranges (B) to the range of the mean variation (A); this ratio is greatest in March and October. In the five months of May to September it is nearly constant, or 1·11; while, in the four months of November to February, it increases from 1·26 to 1·72. From these ratios we may conclude that the diurnal variation obeys nearly a constant law in the months of May to September; while, in the other months, either considerable differences occur in the times of maxima and minima, or the law changes sign, as in March and October.

Column five contains the *hypothetical* ranges of mean curves, when we suppose the agreement of the range of the mean curve with the mean of the daily ranges to be the same in all the months, as in May to September, or that $\frac{B}{A} = 1{\cdot}11$. In this case the ranges for December and January approach in magnitude those for June and July.

Variations for each Day in the Month of March.

We are, as yet, still ignorant of the manner in which the law changes from one form to another in March and October. In order to study this question, I have projected the hourly observations during several months. I shall give here the projections of those only for January, February, and March 1859, a year of considerable disturbance, and for the same months in 1864, a year of slight disturbance. Plates XLII. and XLI.

An examination of these curves will show that the law of diurnal variation does *not* pass regularly from one form to another in the month of March, either by a sliding transfer of maximum to minimum, or by a diminution of movement till the inversal takes place. On the contrary, there appears an utter absence of continuity in any way: the curves are sometimes inverted from one day to the next; sometimes follow in part the law for January, in part the law for June; sometimes resemble wholly the one or the other; frequently resemble neither in any way: and this change appears to occur without order.

The daily curves in March may be distributed into four categories. (I shall include also the conclusions from the months of March 1854 and 1856, the curves for which months I have projected, but which it does not appear to me necessary to give here.)

1*st*, Those curves, which resemble in some degree that for January, having a minimum near 7 A.M., and a maximum between noon and 4 P.M.; to this class the curves for the following days belong:—

1854, March 6, 7, 8 ? 18, 22.
1856, „ 3, 4, 5, 6, 7, 19, 20, 21, 24, 29.
1859, „ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 16.
1864, „ 1, 2, 4, 5, 10, 15.

2*d*, Those resembling the curve for June, in having the maximum about 7 A.M., and a minimum near noon. To this class may be given the curves for

1854, March 14, 16 ? 23, 24, 25, 27.
1856, „ 11, 13, 15 ? 18, 31.
1859, „ 19, 26, 28, 29, 31.
1864, „ 7, 8 ? 11, 12 ? 17 ? 18 ? 25, 26, 29, 30, 31.

3*d*, Those which approach the second class, but have the maximum near 10 or 11 A.M., and minima near 6 A.M., and 1 to 4 P.M., namely,—

1854, March 9, 10, 11, 12, 13, 21.
1859, „ 22, 25, 26, 27 (also April 1, 2, 3, 4, 5, 7, 8).
1859, „ 11 ? 12 ? 13, 14, 17, 18, 25 ? 30, 31 ?
1864, „ 17 ? 19, 21, 23, 24 ?

4*th*. Those which resemble none of these, and happen on the remaining days of each month (not including Sundays, when there were no observations).

The curves resembling that for January occur chiefly towards the beginning of the month, and those resembling the curve for June in the latter part of the month. In some years (as in 1856) the curve preserves a maximum about 10 A.M., before passing to the maximum at 7 A.M. There is no evidence of a gradual shift or inversal (excepting when the mean movement deduced from several days' observations is considered). The law is sometimes partially or wholly inverted in one day—as in March 22, 23, 1854: March 11, 12; 13, 14; 29, 31; 1856: March 19, 21; 1859: and March 5, 7; 8, 9; 18, 19; 24, 25, 26; 28, 29; 1864.

The curves are so irregular and variable in their forms as to prove that *the electric currents producing these motions are in a state of disturbance, and flow in various different directions, from day to day, in the month of March.*

It appeared to me of importance to examine to what extent the form of the mean curves, in the months from November to February, represented the usual diurnal law, and whether the inflexion or slight minimum apparent in these curves near noon was observable in the ordinary daily movement, or was an arithmetical result like the mean curve for March, and due to a combination of different forms. For this end I projected the curves for each day in January and February 1859 and 1864 (see Plates XLII. and XLI.)

Variations for each Day in January and February 1859 *and* 1864.

Commencing with 1864, a year of small disturbance (see Plate XLI.), the minimum of easterly declination occurs generally in the months of January and February at $6^h\ 30^m$ or $7^h\ 30^m$ A.M.; rarely as early as $5^h\ 30^m$ or as late as $8^h\ 30^m$. This regularity of the hour of minimum does not hold for the hour of maximum, which varies betwixt $10^h\ 30^m$ A.M. and $5^h\ 30^m$ P.M. On eight days the maximum occurred near 10 A.M.; on nine days, near noon; on fourteen days, near 2 P.M.; on eight days, near 4 P.M; and on nine days, after 5 P.M. In the mean curve for the month of January 1864, the inflexion near noon is scarcely visible; the maximum is near 2 P.M. In the mean curve for February of the same year, the slight minimum before noon is better marked.

An examination of the several daily curves shows that one-half at least have a minimum or marked inflexion near noon; and such inflexion or minimum does not happen so frequently at any other hour, not an hour of minimum. Although, then, the *mean* curve after the 7 A.M. minimum is a result of different variations, the inflexion or slight minimum observable in the mean curve near noon is probably the representative of a real and persistent phenomenon; and this will appear more evident hereafter.

If we now examine the curves for the same months in 1859, a year of considerable disturbance (see Plate XLII.), and in the mean curves for which there is a secondary minimum at $11^h\ 30^m$ A.M. (see 5th division of Plate XLIII.), we shall

perceive much greater differences in the daily movements. In January 1859 the minimum occurs in the first fortnight betwixt 6^h 30^m A.M. and noon, returning again to $6^h 30^m$ A.M. to reappear near noon at the end of the second fortnight. A similar change of hour of minimum is shown in the latter half of December 1858, the curves for which I have also projected (Plate XLII.) The maximum also happens betwixt 9^h 30^m A.M. and 4^h 30^m P.M., the curve becoming nearly inverted about the 14th and 28th of January. This inversal also, it will be seen, does not take place suddenly, but gradually, and without the irregularity shown in disturbances in high north latitudes, as at Makerstoun. Similar, though less marked, differences of movement appear in the curves for February 1859.

Lunar Action: sometimes as great as that of the Sun.

This curious variation of the form of the diurnal curve is *due chiefly to the action of the moon.* Hitherto the lunar action has been supposed so small, compared with the solar action, that it has been concluded we might neglect the former in considering the laws of the latter, and that the lunar action could be made visible only in combinations of masses of observations, as a residual quantity nearly of the second order. It will be found from what follows, that the lunar action is *sometimes as great as, if not greater than,* the solar action.

I shall consider more minutely, on another occasion, the subject of the variations of magnetic declination due to the action of the moon, and shall then state the methods employed by me to obtain the results. It will suffice at present to say, that, assuming the daily curves to be due to solar and lunar actions superposed, if we subtract from each daily curve the mean solar curve of the corresponding month, the remainder may be supposed due to the lunar action, and to other causes, irregular or regular (such as a variation of the solar action). As the monthly mean curves are undergoing a gradual change, those corresponding to the middle of *each* week were obtained, and these mean curves were subducted from the daily curves in the respective weeks. The results for the lunar month, December 16, 1858, till January 15, 1859—a month, showing a marked lunar action, have been projected, Plate XLIII.

I have given—*1st,* The daily curves (see Plate XLII.); *2d,* The differences obtained as above, under their respective solar hours; and, *3d,* The differences under their respective lunar hours (see Plate XLIII.)—that is, the moon's nearest hour angle when the observation happened, the latter representing the lunar daily curves, with the addition of irregular or other effects.*

These projections will explain, to a great extent, the curious variations of the epochs in the solar diurnal curves which the curves of differences resemble, excepting that they show, more distinctly, the movement of maximum and minimum

* As the moon takes more nearly 25 solar hours to return to the same meridian, it is assumed that there are 25 lunar hours in 360 degrees.

with the position of the moon. When we examine the differences projected under the lunar hours, the law of lunar action is seen more clearly; and it appears with much regularity, considering that irregular solar and other actions are included. In order to diminish the effect of these irregularities, I have taken the means, and projected—1*st*, The mean of the first six curves, those for the days, December 16^d 8^h,1858, till 23^d 14^h, when the moon had the greatest north declination; 2*d*, The mean of the next five, when the moon was near the equator, going south (Es); 3*d*, Of the next six curves (Dec. 29^d 19^h till January 6^d 1^h), when the moon was farthest south; 4*th*, Of the following five curves, when the moon was near the equator, moving northwards (En); and, 5th, The mean of all the curves in the lunation (see fifth division of Plate XLIII.)

The latter presents two nearly equal maxima and two nearly equal minima; the former near the moon's superior and inferior passages of the meridian, and the latter about six hours before and after. When the moon was farthest north, the maximum near the inferior passage was the greater, the minimum near the moon setting being most marked. With the moon near the equator going south, the maxima were nearly equal, but the moon set minimum was by far the most marked. When the moon was farthest south, the maximum near the superior passage was the greater, and the minima were nearly equal. With the moon near the equator going north, the maximum near the superior passage was the greater, and the minimum near moon-rise by far the most marked.

These results, derived from a single lunation, agree almost exactly with those which I had previously deduced from six years' observations.* The dotted curve for the moon near the equator, moving north, including the observations two days later, or to 14th January, represents better the conclusions previously obtained for this position of the moon, in having nearly equal maxima.

These changes of the law of lunar diurnal variation of magnetic declination understood, it will be perceived that the daily curves (second and fourth divisions of Plate XLIII.) in general follow the law and its changes with considerable fidelity.

The range of the solar diurnal curve for December 1858 and January 1859 is nearly 2′·2. The ranges of the mean curves (December 16th 1858, till January 12th 1859) are for the moon:—

Farthest north, ...	= 1′·70
Near the equator, going south, . . .	= 1′·95
Farthest south, ...	= 1′·90
Near the equator, going north, . . .	= 2′·70

I believe that we may conclude, as I have already stated, that the effect of the lunar action is sometimes greater than that of the solar action in the diurnal

* See "Proceedings of the Royal Society of London," vol. x. pp. 482–3.

variation. This is a fact that must be considered in any theory which pretends to account for the variations of the declination magnet.*

It may be supposed that the variable movements in the month of March also are, to some extent, due to the lunar action which shifts its epochs from day to day with the hour for which the moon is on the meridian; and this is no doubt the case; but a first approximation to eliminate the lunar effect from the solar curves has shown me that the irregularity remains after the elimination nearly as before. I shall endeavour hereafter to present the solar curves nearly free from the lunar effect.

I have stated elsewhere, that the lunar law also passes through a period of inversion, near the equator, like the solar law, when the sun passes from one hemisphere to another.† This dependence of the law of lunar action upon the position of the sun complicates the phenomena, and an effort for their separation will be best made when the lunar laws have been investigated.

Diurnal Variation at different Latitudes.

In the preceding pages we have examined the change of law of the diurnal variation of magnetic declination, with the period of the year at a single station near the magnetic equator. It will be desirable now to investigate the mode in

* I have been obliged here to enter upon laws which will be the subject of another paper, in order to explain certain apparent irregularities of solar action. I cannot leave the subject at present without suggesting that the variations of magnetic declination are probably due to currents in an electro-sphere, which, it appears to me, must surround each heavenly body. I have suggested elsewhere that the solar spots are due to disruptions of these currents within the solar atmosphere (below the photosphere, or forming part of the photosphere itself), produced by the planets, and depending for their number and magnitude on the position (latitude and distance) of the latter relatively to the plane of the solar equator. (See a letter addressed to Sir DAVID BREWSTER, December 21st 1857, published in the "Philosophical Magazine," July 1858.) If this idea has any basis, we may suppose it probable that the moon exercises some similar action upon the earth's electro-sphere, an action depending for its amount on the electric tension of the spheres, these again depending upon that of the sun. Some such idea is necessary to explain the fact, that the lunar effect is very variable in its amount, varying from a movement of the free needle of about 5′, in some days, of one lunation, to a tenth part of this movement in succeeding lunations.

† See "Proceedings of the Royal Society of London," vol. x. p. 1482, 861. I may note here that, as I showed in the "Makerstoun Observations for 1844" (Trans. Roy. Soc. Edin., vol. xviii. p. 354), the difference of the law of solar diurnal variation of declination in Europe for summer and winter was of the same kind as that betwixt the laws for the same periods near the equator, where the movement is inverted; so it followed from the inversion of the lunar law near the equator, with the sun in the northern and southern hemispheres, that in high latitudes, the lunar law should present a greater range in summer than in winter, and that the mean law in high north latitudes should be nearly the opposite of that for high south latitudes. The latter fact should have been evident from preceding observations in mean latitudes of the two hemispheres, and on the discovery of the inversal of the lunar variation with the sun's passage of the equator, I examined these observations for this end. Unfortunately the discussion of the Toronto Observations by General Sabine gave a result nearly the inverse of those derived from the observations at Makerstoun and Prague in the same hemisphere, as I pointed out in the note cited above. General Sabine then discovered that west had been substituted for east in the discussion of the Toronto Observations; and this correction made, the approximate opposition of the laws for the two hemispheres was at once evident. Of the former fact I satisfied myself by a rediscussion of the "Makerstoun Observations" rejecting the large disturbances; and it has been verified also by the late Mr BACHE, from observations at Philadelphia, though only two years after the result deduced by me of the inversion of the law at the equator had made the conclusion, if not certain, at least extremely probable.

which the law changes with the latitude in each month of the year. For this purpose it will be preferable to reduce the movements of the horizontal needle to those of the needle freely suspended in the direction of the magnetic inclination, since the latter become greatly exaggerated in high latitudes when observed in the more commodious suspension of the horizontal needle.

Stations.

Ten stations have been chosen for this investigation. The following table contains the facts required concerning them :—

TABLE III.—*Data relating to the Stations and Observations of Magnetic Declination considered in this Paper.*

Station. No.	Station. Name.	Latitude.	Longitude.	Height above Sea.	Magnetic Inclination.	Years of Observation.	Directors of the Observatories.
1	Makerstoun,	55° 35′ N.	0h 10m W.	Feet. 213	71° 30′ N.	1843–46.	Mr J. A. Broun.
2	Toronto, .	43 40 N.	5 17 W.	...	75 15 N.	1842–48.	Lieutenant Riddell, R.A., Lieutenant Lefroy, R.A., and Lieutenant Younghusband, R.A.
3	Simla, . .	31 6 N.	5 9 E.	8000 ?	41 40 N.	1842–44.	Major-General Boileau.
4	Bombay, .	18 56 N.	4 51 E.	A few feet.	18 44 N.	1851–55. 1857–61.	Captains Montriou and Ferguson of the Indian Navy.
5	Madras, .	13 4 N.	5 21 E.	A few feet.	7 40 N.	1846–50.	Mr T. G. Taylor, Colonel Worster, and Captain Jacob.
6	Trevandrum,	8 31 N.	5 8 E.	200	2 30 S.	1853–64.	Mr J. A. Broun.
7	Singapore,	1 19 N.	6 56 E.	A few feet.	12 40 S.	1841–45.	Captain C. M. Elliot.
8	St Helena,	15 57 S.	0 23 W.	1760	22 0 S.	1842–47.	Lieutenant Lefroy, R.A., and Lieutenant Smythe, R.A.
9	Cape of Good Hope, .	33 56 S.	1 14 E.	A few feet.	53 20 S.	1841–46.	Lieutenant Eardley Wilmot, R.A.
10	Hobarton,	42 53 S.	9 50 E.	105	70 20 S.	1843–48.	Commander Kay, R.N.

1. Transactions of the Royal Society of Edinburgh, Vols. XVII. XVIII. XIX.
2, 8, 9, 10. Observations at the Colonial Observatories, published under the superintendence of Major-General Sabine.
3. Simla Observations. Copy of Abstracts issued from the Simla Observatory.
4. Bombay Observations, printed at Bombay by order of Government.
5. Madras Observations, printed at Madras by order of Government.
6. Observations, unpublished.
7. Singapore Observations, printed by order of Government.

It will be perceived that five of the stations have north magnetic latitude, and five south magnetic latitude, while their distribution varies in longitude betwixt $5^h\ 17^m$ W. and $9^h\ 50^m$ E. The means are not derived from observations in the same years, nor from the same number of years' observations, and they are not therefore strictly comparable. It is conceived, however, that in each

case there is a near approximation to the *mean* diurnal variations for each month at each station, so near that the addition of any number of years' observations would not alter in any way the conclusions of this paper.

Comparison of the Mean Diurnal Variations at different Stations.

The monthly mean diurnal variations for each station having been reduced to those of the needle freely suspended in the direction of the magnetic inclination, the resulting values were projected in curves, Plate XLIV., which we shall proceed to consider month by month.

January.—The minimum of *easterly* declination (north end of needle farthest *west*) occurs from Bombay to Hobarton at beween 7 A.M. and 9 A.M. At Simla there is a minimum at 7 A.M., but rather less marked. This minimum evidently diminishes gradually in importance as we proceed north, being shown at Makerstoun in a very faint degree only. This minimum is followed by a maximum occurring betwixt 8^h and $9^h\ 30^m$ A.M., which also becomes less marked as we proceed towards Makerstoun, where it appears as an inflexion on the descending branch of the curve; it disappears in like manner south of Trevandrum, where it is seen as a slight maximum on the ascending branch. The minimum near 8 A.M. can, therefore, be traced from Hobarton to Makerstoun, while the maximum near $9^h\ 30^m$ A.M. appears only north of the equator. From Makerstoun to Trevandrum the maximum is followed by a minimum (the principal minimum from Simla northwards) betwixt $1^h\ 30^m$ P.M. and noon. This minimum, which is so well marked from Madras northward, almost disappears in the next $4°\frac{1}{2}$ of latitude, being seen faintly at Trevandrum. At Singapore it is no longer visible.

At Trevandrum and the stations south of it, the principal maximum of easterly declination is attained near 2 P.M., excepting at St Helena, where it occurs near noon. As marked, a displacement occurring betwixt Singapore and St Helena as betwixt Singapore and Madras. In the latter case, the mode in which the transition happens is shown by the curve for Trevandrum; as there is no station betwixt Singapore and St Helena, the leap appears more abrupt.

A secondary maximum and following minimum occur in the group Simla to Trevandrum at 4 P.M. and 6 P.M., which become more marked in succeeding months. St Helena and the Cape have a minimum at 5 and 6 P.M., followed by a maximum near 9 P.M. At Makerstoun, the station farthest north, the maximum occurs near 10 P.M.

February.—The remarks made upon the curves for the preceding month apply generally to those for this month. The curve for St Helena differs less from those for Singapore and the Cape than before the maximum occurring betwixt 1 and 2 P.M.

March and April.—In these months, when the sun is near the equator, the

opposition betwixt the two hemispheres is well seen. The northern curves have the maximum betwixt 8 and 9 A.M., and the minimum near 1 P.M. The southern curves have the minimum betwixt 9 to 10 A.M., and the maximum betwixt 1 and 2^h 30^m P.M.; noon at St Helena in April. The passage from one form to another is shown at Trevandrum in March, and at Singapore in April. At St Helena, also, a change has begun, which throws the maximum to noon. The maximum and minimum in the highest north latitudes occur about one hour earlier than the minimum and maximum at the stations farthest south. The passage from the northern to the southern form does not appear in these months to take place by a sliding shift of the epochs; at Trevandrum the curve for March approaches the straight line, and is very nearly that which might have been derived from the mean of the curves for Madras and Singapore in the same month.

May, June, July, and August.—These months may be considered together, as the comparison gives nearly the same conclusions. The first remark is, that when the sun is farthest north, there is really no inversion of the law of movement betwixt the latitudes of 56° N. and 43° S. The movements are, with slight exceptions, as follow:—The north end of the needle moves eastward from after midnight till near 7 A.M., it then turns westward, attaining the extreme westerly position at from 1^h to 10^h 30^m or 11^h P.M., the former hour farthest north, the latter farthest south, and near noon at the equator. The north end of the needle then moves eastwards till from 3 to 6 P.M., the latter hour at the northern stations, the former at the southern, and near 4^h 30^m P.M. at the equator. The north end then moves westerly for a short period at all the stations, from Simla southwards, or continues slowly eastwards at the most northern stations. From about 8 P.M. the movement is easterly at all the stations, excepting Hobarton, where the easterly movement does not commence till near midnight.

In general, the curves for these months show a slight shift of the epoch of minimum, and at Hobarton the afternoon maximum becomes the principal one; but there is no inversal of the law of movement.

The curves for St Helena show a curious movement near noon, which appears in some degree the equivalent of the movement at Simla, Bombay, and Madras, in November to February. Thus, at St Helena, in May, June, July, and August, there is the maximum at 7 A.M., a minimum at 10 A.M., a maximum at noon to 3 P.M., and minimum at 5 P.M.; while at Bombay, &c., in November, December, January, and February, there is the minimum at 7 A.M., a maximum at 9^h 30^m A.M., a minimum at noon, and maximum at 4 P.M. It seems probable that at some station betwixt St Helena and the equator this resemblance of opposition would be better shown.

September.—In this month the minimum, which occurs betwixt noon and 1 P.M., from Trevandrum northwards shifts suddenly to 10 A.M. at Singapore, at which hour it occurs at all the southern stations, so that the shift occurs in this

case between 8°½ N., and 1°½ N. The St Helena curve resembles the inverse curve for Simla and Madras in March.

October.—The opposition of the curves for the two hemispheres begins to show itself again nearly as in the curve for April, the Singapore curve for that month resembling the Trevandrum curve for October. The passage from one form to the other is not shown in the same way, nor with the same distinctness, as in March.

November and December.—The curves for these months resemble, generally, those for January and February, the minimum at noon in the Indian curves not being so well marked in December as in the months of November and January. In the winter months the minimum occurs latest at Toronto, which has the highest magnetic latitude.

Ranges of the Diurnal Variations of the Free Needle.

In order to compare the arcs within which the movements of the free needle are performed at the different stations, the following table has been prepared :*—

TABLE IV.—*Ranges of the Mean Diurnal Variations of the Needle freely suspended in the direction of the Magnetic Inclination.*

Months.	Makerstoun.	Toronto.	Simla.	Bombay.	Madras.	Trevandrum.	Singapore.	St Helena.	Cape of Good Hope.	Hobarton.
	′	′	′	′	′	′	′	′	′	′
January,	1·85	1·49	1·22	1·73	1·63	2·06	2·64	3·45	3·08	4·05
February,	2·25	1·54	1·16	1·39	0·96	1·48	2·89	4·81	4·56	4·28
March, .	2·94	2·30	2·69	2·57	2·36	0·79	1·67	4·57	4·35	3·30
April, . .	3·56	2·48	3·88	4·07	3·71	1·67	1·13	3·06	2·80	2·60
May, . .	3·48	3·03	4·27	4·72	4·69	2·90	2·24	2·45	2·34	1·56
June, . .	3·67	3·08	4·37	4·69	4·77	3·06	1·83	3·00	1·93	1·28
July, . .	3·35	3·05	3·90	4·39	4·57	3·06	1·94	3·17	1·99	1·36
August, .	3·42	3·48	4·36	5·42	4·75	3·64	2·60	3·32	2·75	1·87
September,	3·30	2·51	3·99	5·38	5·28	3·31	2·07	2·22	2·56	2·65
October, .	3·01	1·74	2·36	2·59	2·37	1·27	2·38	4·11	3·55	3·80
November,	2·39	1·58	0·98	0·90	1·50	2·14	2·97	3·75	3·81	4·08
December,	1·91	1·33	0·78	1·16	1·17	2·33	2·90	3·28	3·26	4·13
Mean, . .	2·93	2·30	2·83	3·25	3·14	2·31	2·35	3·42	3·08	2·91

Minimum Ranges.

From this table it appears that the minimum movement of the free needle, between the latitudes of 56° N. and 43° S., occurs in January at Simla; it descends to Madras in February, to Trevandrum in March, and to Singapore in April. In May, June, July, and August, the minimum movement occurs at Hobarton (or perhaps between Hobarton and the Cape); in September at Singapore, or between Singa-

* The values in the table are taken from the reduced hourly observations, and not from the curves, which would give the ranges slightly greater in some cases.

pore and St Helena; it happens in October at Trevandrum; in November at Bombay, and in December again it has returned to Simla. The minimum movement of the free needle thus passes from latitude 31° N. (or it may be somewhat north of this latitude) in December and January, to the equator in March and April, and thence to 43° S., in June and July, recurring near the equator in October, and returning to Simla in December.

Maximum Ranges.

The maximum movement of the free needle occurs in January at Hobarton; in February and March, at St Helena (or between St Helena and the Cape); in April, May, June, July, August, and September, at Bombay or Madras (or perhaps betwixt these two stations); in October at St Helena; and in November and December at Hobarton. The maximum movement changes its locality with the period of the year, like the minimum, but it does not proceed so far north; in April and June, perhaps, it approaches Simla. The years from which the Simla results are derived give probably somewhat less than the mean values; those for Madras, on the other hand, give values somewhat too great.

The maximum movement of the free needle never occurs near the equator, as at Trevandrum and Singapore, but passes suddenly, during the minimum movements at these stations, from Bombay to St Helena, or from St Helena to Bombay. The greatest movement of the free needle at any station occurs at Bombay in August and September, and is equal to 5′·42, which is equivalent to a mean diurnal oscillation at Toronto of the horizontal needle of 20′·9; while the greatest monthly mean diurnal oscillation of the horizontal needle at Toronto, deduced from the observations for 1842 to 1848, including the disturbance years of 1847–48, is only two-thirds of this quantity. In whatever way we combine the results to represent the mean movement in each hemisphere, it seems to me probable that the mean movement of the free needle for the *whole earth* is greatest in August.

The smallest movement at any station occurs at Simla and Trevandrum; at the latter from a combination of different movements. The least and greatest movements of the free needle occur nearly under the tropic of Cancer.

When we take the mean of the monthly ranges for each station, we find Toronto to have the smallest mean, and next the equatorial stations of Trevandrum and Singapore.*

* It should be remembered, that we are examining the ranges of the monthly mean diurnal variations, and that these are diminished near the equator, by the change of law occurring in months near the equinoxes.

Change of Epochs of Maxima and Minima.

In order to study the change of epochs of the minima and maxima of the ranges of the mean diurnal variation of magnetic declination at each station, we may divide the stations into two groups—the extra tropical and the intertropical. It will be at once evident, that the epochs of minima and maxima for the northern groups have the same, or nearly the same, position relatively to the December and June solstices, that those for the southern groups have to the June and December solstices.

Thus Makerstoun, Toronto, Hobarton, and the Cape, have only one minimum and one maximum of range. At Makerstoun they occur near the December and June solstices; at Hobarton near the solstices of June and December, respectively. At Toronto the minimum occurs near the December solstice; at the Cape near the June solstice; while at Toronto the maximum range happens two months after the June solstice, it occurs two months after the December solstice at the Cape of Good Hope.

A similar resemblance of opposition may be traced betwixt the northern and southern intertropical stations. Thus, at Simla, Bombay, Madras, and Trevandrum, the minima of range occur in October or November, and in February or March; while at Singapore and St Helena they occur in April or May, and in September. That is to say, in the northern stations one or two months before the December solstice, and near the vernal equinox; while at Singapore and St Helena they happen one or two months before the June solstice, and at the autumnal equinox. In like manner, the maxima occur in the stations from Bombay to Trevandrum in August or September, with the secondary maximum in January or December; while at Singapore and St Helena the secondary maximum occurs in August, and the principal maximum in February.*

General Remarks.

When we consider the curves for each station separately, we perceive that they pass gradually from one form to another from month to month—the changes being generally a gradual shift of the critical hours, and a variation of the relative values of maxima and minima. The chief exception to this remark is to be found in the change at Trevandrum in the month of March. In the Indian group, and at stations even farther north, which I have not considered in this paper, the

* At all the stations there is a kind of double in the epoch of principal maximum, seen more or less markedly in May or June, and in August, in the northern stations; in October or November, and in February, in the southern stations; at Singapore the range for November is slightly greater than that for February. I should also notice, as a deviation from the law of change of minimum from one station to another, the occurrence of the minimum oscillation at Madras in December, instead of November, as its position betwixt Bombay and Trevandrum would indicate. I feel inclined to believe that this deviation would not have appeared had the ranges for Bombay and Madras been derived from observations during the same years.

curves present an apparent anomaly in the movement betwixt 7 A.M. and noon for the four months November to February, which is a combination of the movements in high north and south latitudes at the same period. Yet we find that this form continues in a general way throughout the year, the maximum shifting from $9^h\ 30^m$ A.M. to $7^h\ 30^m$ A.M.

It appears from the first part of this paper, that the law of movement varies with great rapidity from day to day near the magnetic equator about the time of the equinoxes, and especially in the month of March, when the mean curve at Trevandrum approaches the straight line. From the second part it appears that the law of movement varies rapidly with the latitude at the same time of the year near the equator. There is a zone of disturbance near the equator when the sun passes from one hemisphere to another. It will require investigations at other stations to determine how far this zone extends. If we may judge from the epochs of minimum movement, it would appear that the zone is north of the equator when the sun is in the southern hemisphere, that it reaches the equator with the sun when the disturbance has the greatest intensity, and that it passes south of the equator as the sun moves northwards.

The number of stations should be increased betwixt the latitudes of St Helena and Singapore, to show clearly the mode in which the law changes near the equinoxes from one station to another, though there can be little doubt that a station midway betwixt the two would show movements represented nearly by the mean of the curves for St Helena and Singapore. I have myself made observations of magnetic declination simultaneously at Trevandrum, at a station ninety miles north on the magnetic equator, and at Cape Comorin, forty miles south of Trevandrum, during the equinoxes. I shall hereafter show the incremental curves due to these changes of latitudes.*

I have already noticed a fact pointed out by me in the Makerstoun Observations for 1844,† that the difference of ordinates of the curve representing the summer movement at Makerstoun, and of that representing the winter movement at the same place, gave a curve representing the typical movement of the southern hemisphere; so that the relation of the summer curve at Makerstoun to the winter curve at the same station, was the same in kind as that betwixt the two opposite curves seen near the equator. In other words, the summer curve in high north latitudes was only diminished in range by the change which sufficed to invert the curve for the same epoch at the equator.‡

* Observations were also made at a height of 6000 feet above Trevandrum, and twenty miles W.N.W. of it.

† Trans. Roy. Soc. Edin. vol. xviii. p. 354.

‡ It is not meant that the change was the same in amount in high latitudes and near the equator, though, when we consider the variations of the horizontal needle at Makerstoun and Trevandrum, the changes are nearly equal.

If we subtract the curve with maximum movement from that of minimum movement of the *free* needle at each station in the higher latitudes where the curves follow nearly the same law, or

This idea has since then (I have no doubt without any knowledge of my note on the subject) been generalised in several discussions by General SABINE,* who, in subtracting the ordinates of the mean curve for the whole year at any station from the mean curves for the summer and winter half years, has obtained two curves opposed to each other, representing the changes produced in the mean curve by the sun in each hemisphere.

It appears to me that this fact does not merit any greater value than that attributed to it in the original note already cited; since if we represent the two curves for the half years when the sun is north and south of the equator by the equations

$$y_n = a_0 + a_1 \sin(\theta + c_1) + a_2 \sin(2\theta + c_2) + \ldots$$
$$y_s = b_0 + b_1 \sin(\theta + d_1) + b_2 \sin(2\theta + d_2) + \ldots$$

The curve for the mean of the year will be represented by the equation

$$y = \frac{a_0 + b_0}{2} + \frac{a_1 \sin(\theta + c_1) + b_1 \sin(\theta + d_1)}{2} + \ldots;$$

and the curves representing the difference betwixt that for each half year and the mean of the year will be given by the equations

$$y_n - y = \frac{a_0 - b_0}{2} + \left(\frac{a_1}{2} \sin(\theta + c_1) - \frac{b_1}{2} \sin(\theta + d_1)\right) + \ldots$$

$$y_s - y = -\frac{a_0 - b_0}{2} - \left(\frac{a_1}{2} \sin(\theta + c_1) - \frac{b_1}{2} \sin(\theta + d_1)\right) + \ldots;$$

or the two equations will have the form

$$y_n - y = A_0 + A_1 \sin(\theta + C_1) + A_2 \sin(2\theta + C_2) + \ldots$$
$$y_s - y = -A_0 - A_1 \sin(\theta + C_1) + A_2 \sin(2\theta + C_2) + \ldots;$$

equations to two similar curves with opposed ordinates. When c_1 and d_1 do not differ much, which is generally the case, the singular points in the difference curves will occur nearly at the same time as in the curves for the whole year; or $C_1 = \frac{c_1 + d_1}{2}$ approximately.

This, however, is true of any curves varying in range, as in the diurnal curve of temperature for example, which obeys a law resembling that of the magnetic

the two curves of maximum movement from each other where they have opposite forms (as the curve of August from that of December at Trevandrum, and that of August from the curve of February at St Helena), we shall find that the change of movement, or the range of the difference curve, is three or four times greater near the equator than in high latitudes, St Helena being the station of greatest change of form, the range of the difference curve being seven minutes and a half (7′·5).

If the variation of the movement of the *free* needle from summer to winter had been nearly as great at Makerstoun, Hobarton, or Toronto, as near the equator, the curve would have been completely inverted. It is simply because the variation of range is less in high latitudes than near the equator that the inversion does not take place there. This fact does not appear when we compare the movements of the horizontal needle.

* It has also been employed by the Rev. P. SECCHI in the same way as by General SABINE. Monthly Notices Roy. Ast. Soc. vol. xv. p. 27.

declination in high latitudes, having a greater range in summer than in winter. A similar operation then performed on the curves of temperature would produce a similar result; but, as we know, the temperature curve is not inverted in the southern hemisphere, the maximum temperature occurring somewhat after noon everywhere.

Though this process of subtracting the mean curve from its two parts cannot have any marked value in the consideration of the phenomenon of the diurnal variation of magnetic declination, yet I desire to point out that the subtraction of curves, or of their equivalent equations, from each other, indicated by me in my original note already cited, may lead in other ways to definite ideas of the increments of movement (the curval increments, if I may so term them), which are due in one place to a change of position of the sun, or which are due for the same position of the sun to a change of latitude. These operations, which have been already performed by me for different places, may be of importance with reference to the comprehension, not only of the rapid changes of law near the equator, but also of the changes due to an approach to the magnetic poles.

Ingenious attempts have been made to connect the diurnal variations of magnetic declination with the electrical currents observed by means of telegraph wires. These currents are, I believe, quite local; it is extremely probable that they produce variations of a second order: others of the first order are due to the polar currents, connected with the polar lights; but these are superposed on the regular solar variations. We have, I think, only to look at the group of curves for the months of May to August, to see the fallacy or the insufficiency of all the hypotheses proposed as yet to explain these variations. We have, it appears to me, to deal with a general (in opposition to local), if not a cosmical cause. The position of the station as regards coasts, centre of continents, or as to height, does not seem to produce any marked variation in the regularity of the law and its change from north to south, in the months referred to.

I have already noticed the great effect of the lunar action, sometimes exceeding that of the sun, and have offered a suggestion, which might lead to a theory. I would only add, at present, that since writing the note, page 680, it has occurred to me that I endeavoured to show in 1850, in the General Results of the Makerstoun Observations (I am not able to cite the page at present), that the aurora borealis was most frequent near the epoch of full moon. Though the number of auroræ, from which this conclusion was derived, was too limited to give it great weight, yet, as the auroræ were sought for on the appearance of the slightest irregularity in the movement of the magnets (a sure premonition) it merited a greater weight than one deduced from masses of observations by different observers, made without system, and noted chiefly when most easily seen—that is, when the moon does not shine. I believe I also pointed out the occurrence of

the most brilliant auroræ near the epoch of opposition. These observations and conclusions may give some support to the suggestion I have made as to the action of the moon on the earth's electro-sphere, the tension of which I have supposed to depend upon that of the sun.

I would draw attention again to MAIRAN'S theory, of a connection betwixt the zodiacal light and the solar light. I asked before, "Is the zodiacal light not the magnetic [or electric] ether in a luminous state?" * I have frequently observed the zodiacal light in 8½ N. latitude, at Trevandrum, when, according to the observations, the summit of the lens of light must have considerably surpassed the zenith at sunset. Whatever this light may be, or whatever its cause, it cannot fail to act upon the earth and its satellite, and to be acted on by them. That it extends far beyond the earth's orbit in a non-luminous state, and that it is acted on by more distant planets than the earth, seems to me probable; and it may be, that in this action we should seek the connection betwixt the position of the planets and the solar spots; which spots I have already suggested, are disruptions of the electric atmosphere of the sun,† disruptions passing through the photosphere—the photosphere itself being the dense base of the zodiacal light.

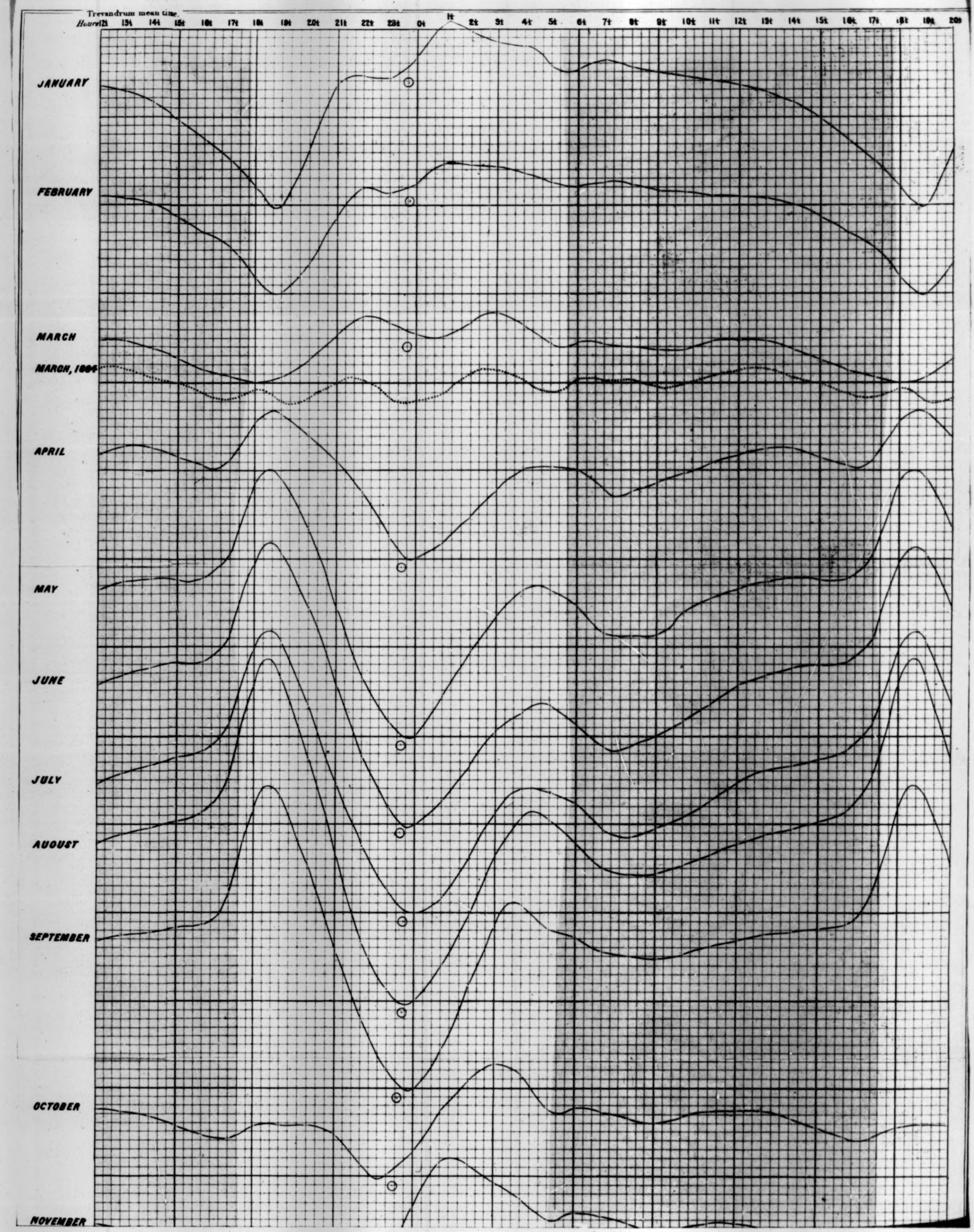
* See Letter, dated Trevandrum, 21st December 1857. Philosophical Magazine, July 1858.

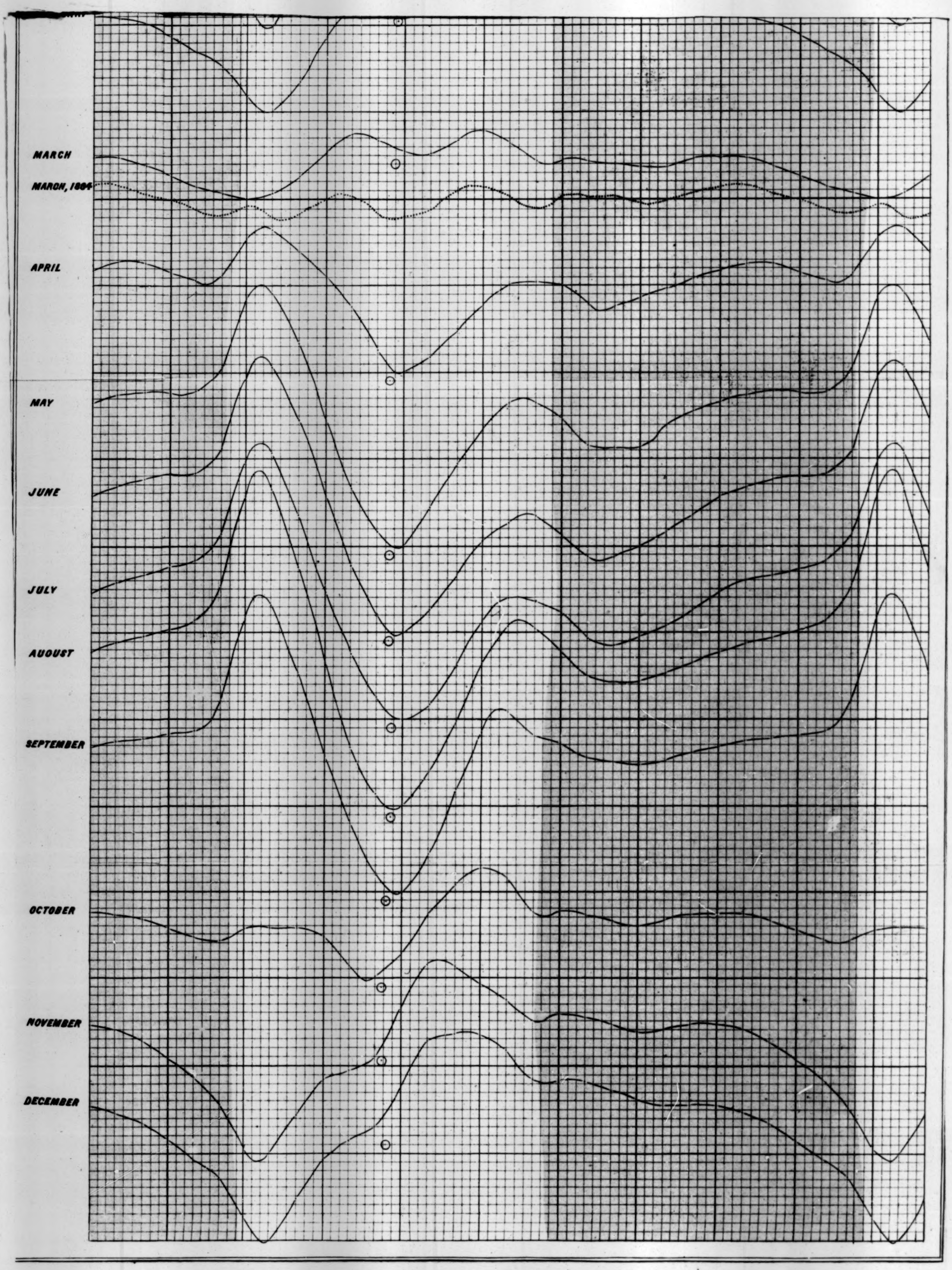
† See Letter just cited, where I have also noticed that CASSINI and MAIRAN supposed a connection betwixt the intensity and extent of the zodiacal light and the solar spots.

PLATE

MONTHLY MEAN CURVES OF THE DIURNAL VARIATION OF MAGNETIC DECLINATION AT TREVANDRUM, DEDUCED FROM 12 YEARS' OBSERVATIONS, (1853-186

BY J. A. BROUN, F.R.S.



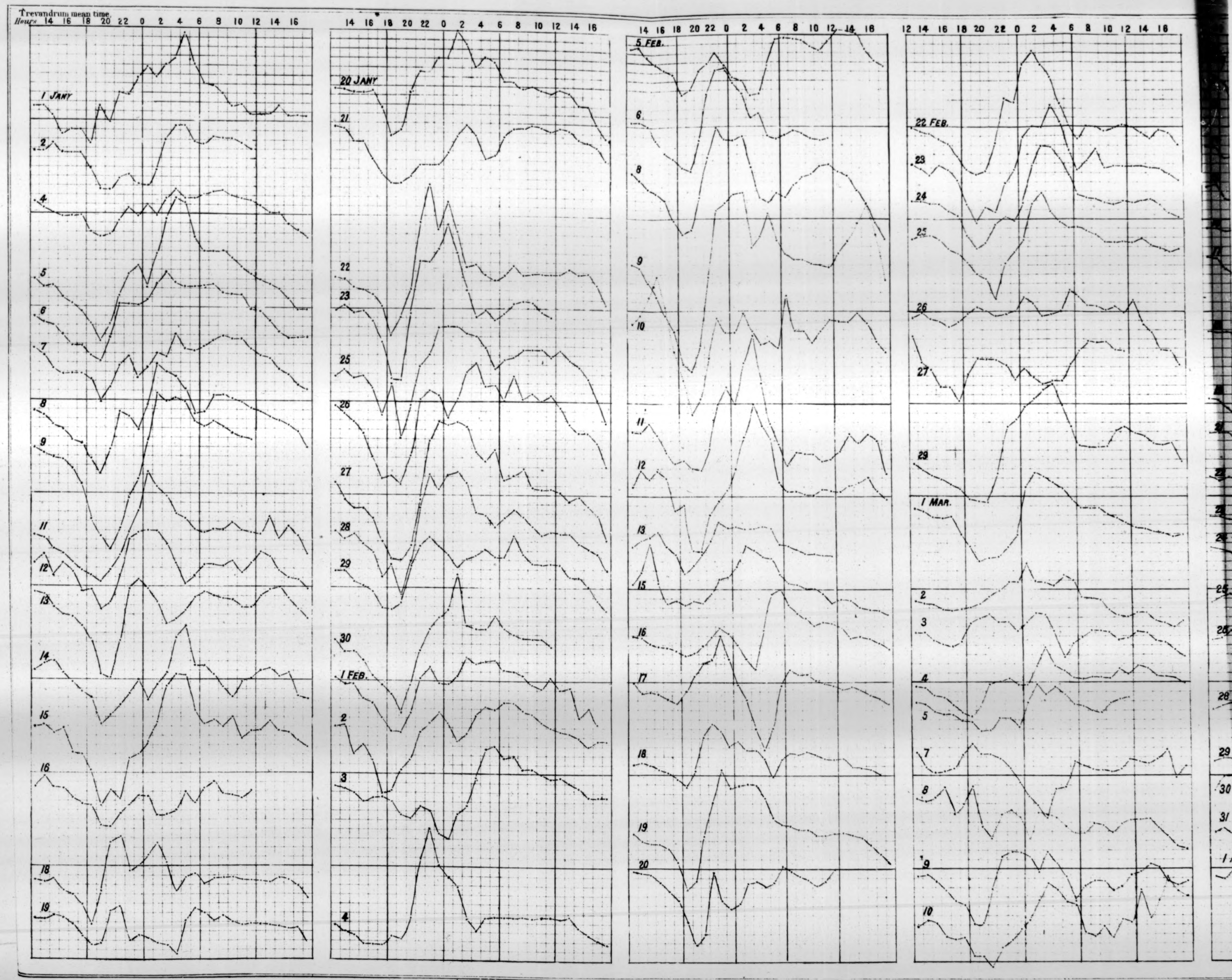


Ascending Curves indicate the movement of the North end of the Needle towards the East. ⊙ *True Noon.*

SCALE: *One inch = One minute of arc.*

The part shaded shows when the Sun is below the Horizon.

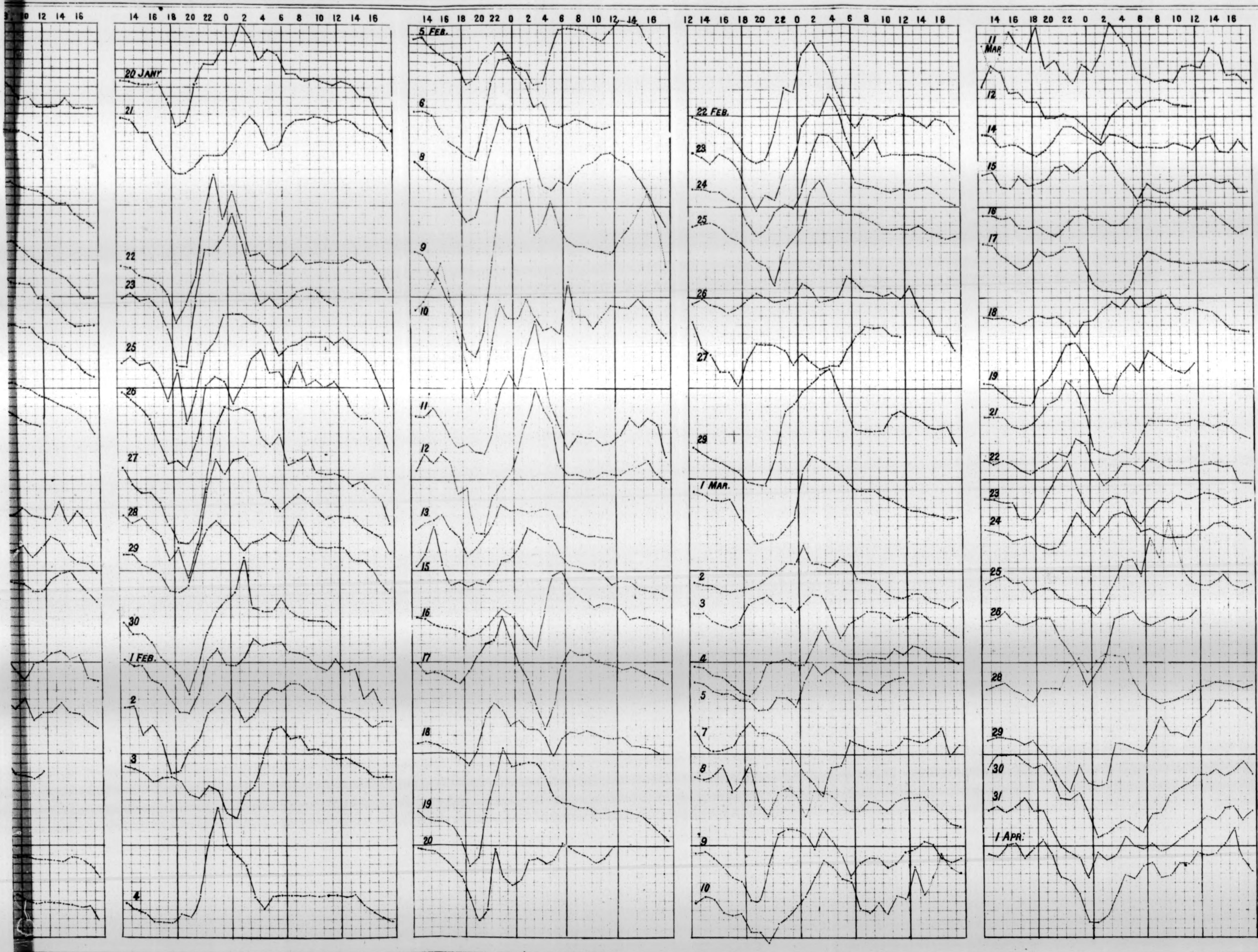
DAILY CURVES OF MAGNETIC DECLINATION, AT TREVANDRUM, IN JANUARY, FEBRUARY, & MARCH, 1864. BY J. A. BROUN, F.R.S.



One division = 0'·25 ; One inch = 2'·5. Ascending Curves indicate movement of the North end of the needle towards the East.

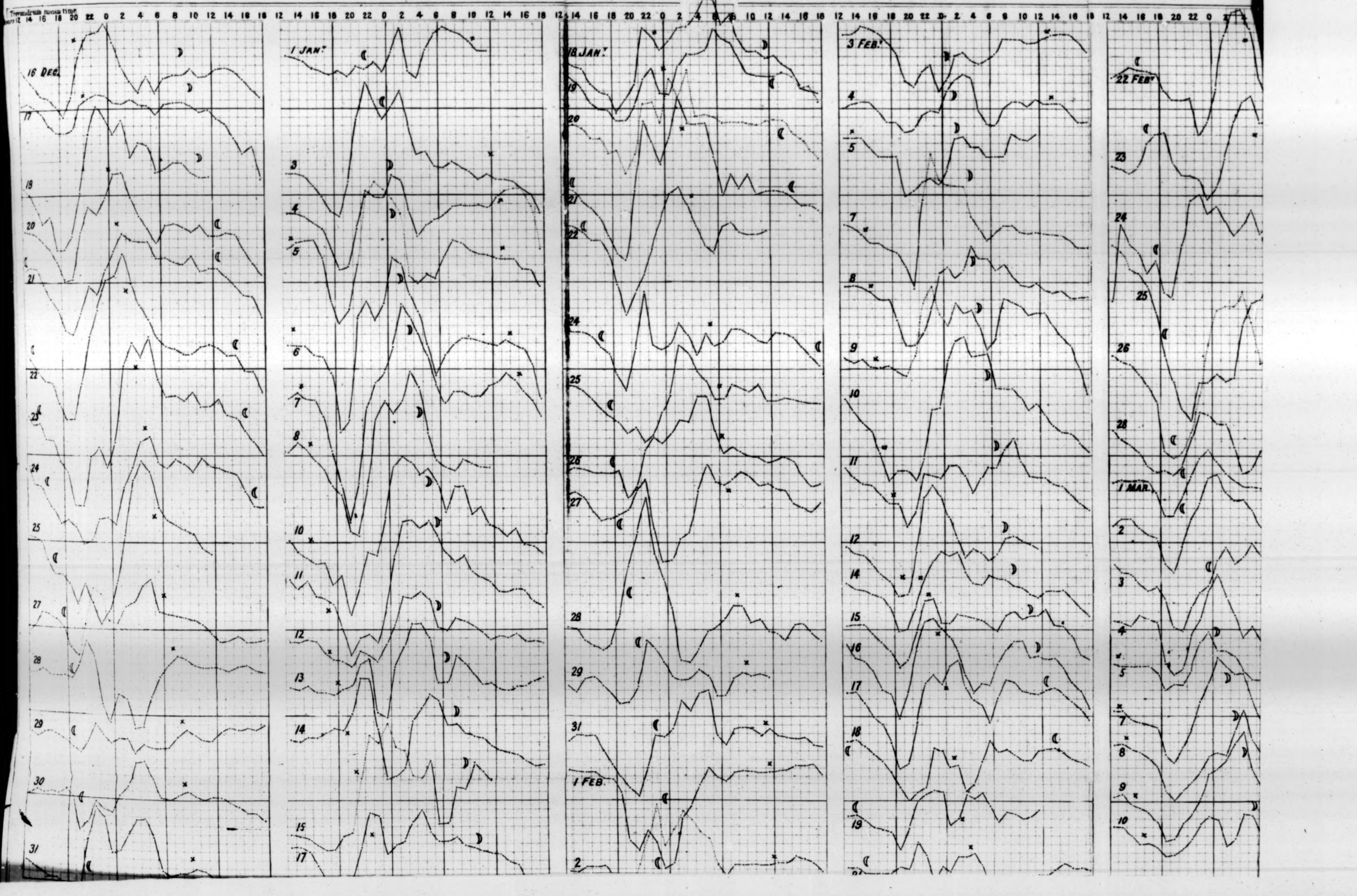
DAILY CURVES OF MAGNETIC DECLINATION, AT TREVANDRUM, IN JANUARY, FEBRUARY, & MARCH, 1864. By J. A. BROUN, F.R.S.

PLATE XLI.



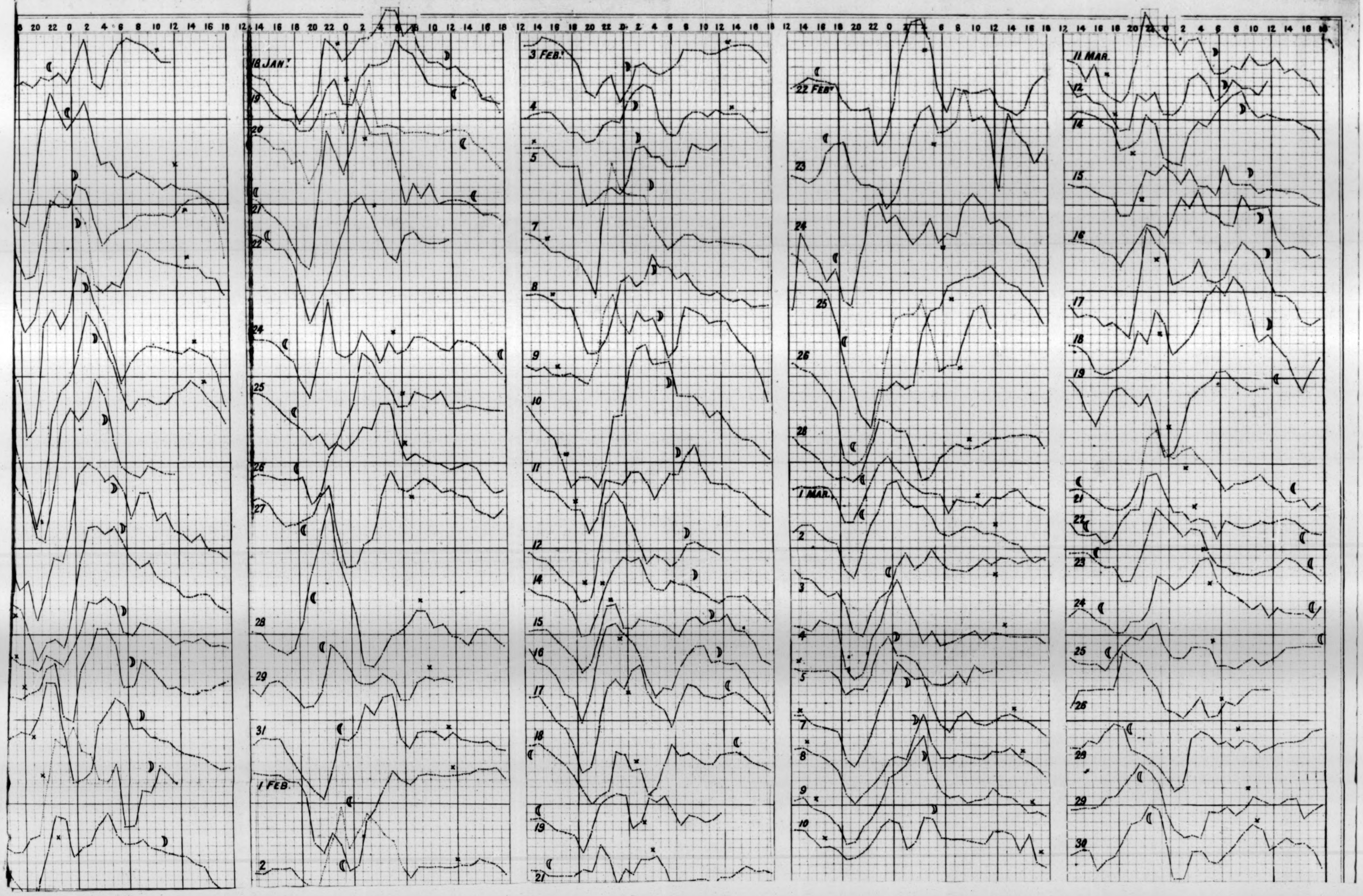
One division = 0'·25 ; One inch = 2'·5. Ascending Curves indicate movement of the North end of the needle towards the East.

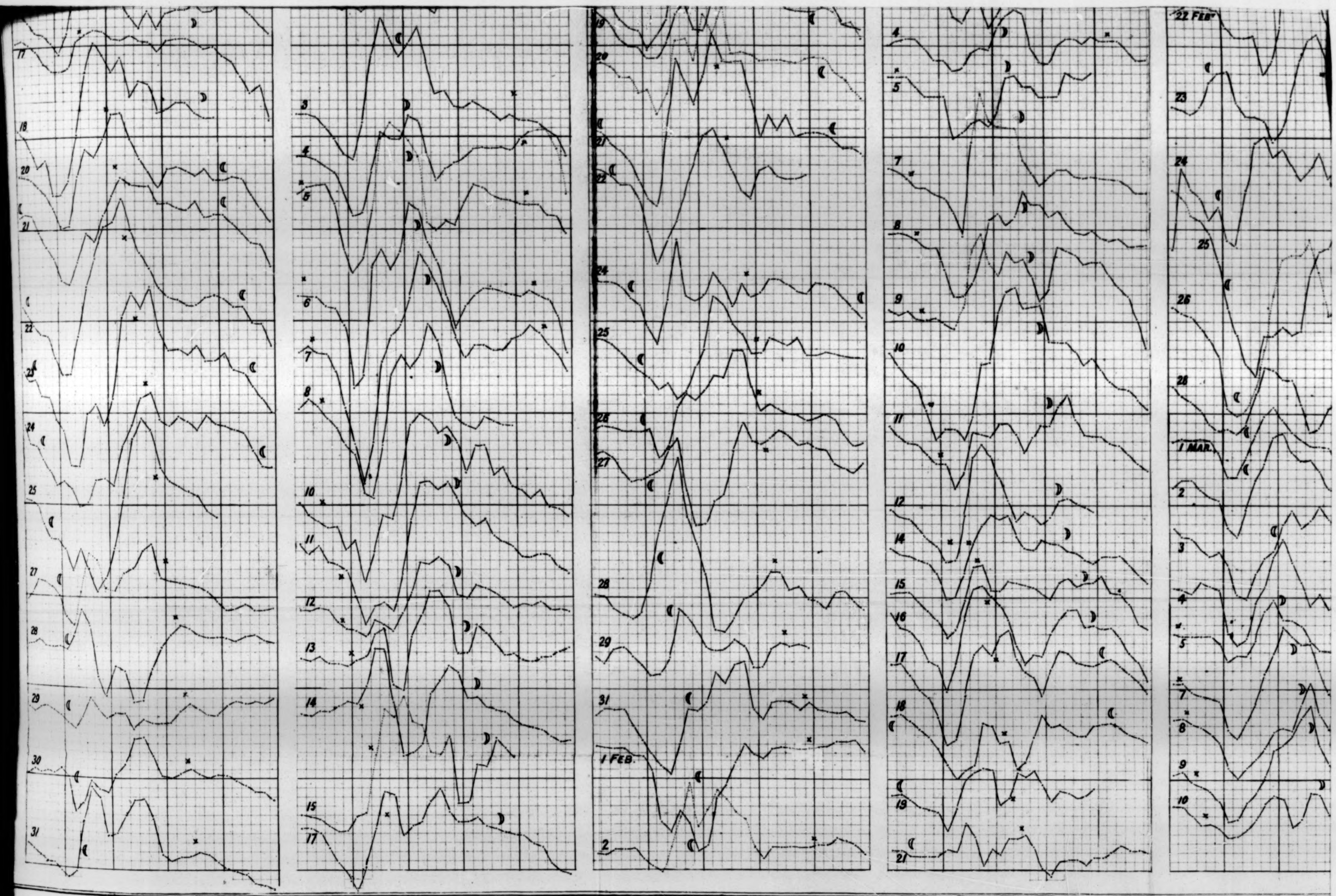
DAILY CURVES OF MAGNETIC DECLINATION AT TREVANDRUM, FROM 18TH DECEMBER, 1858, TILL 30TH MARCH, 1859. BY J. A.



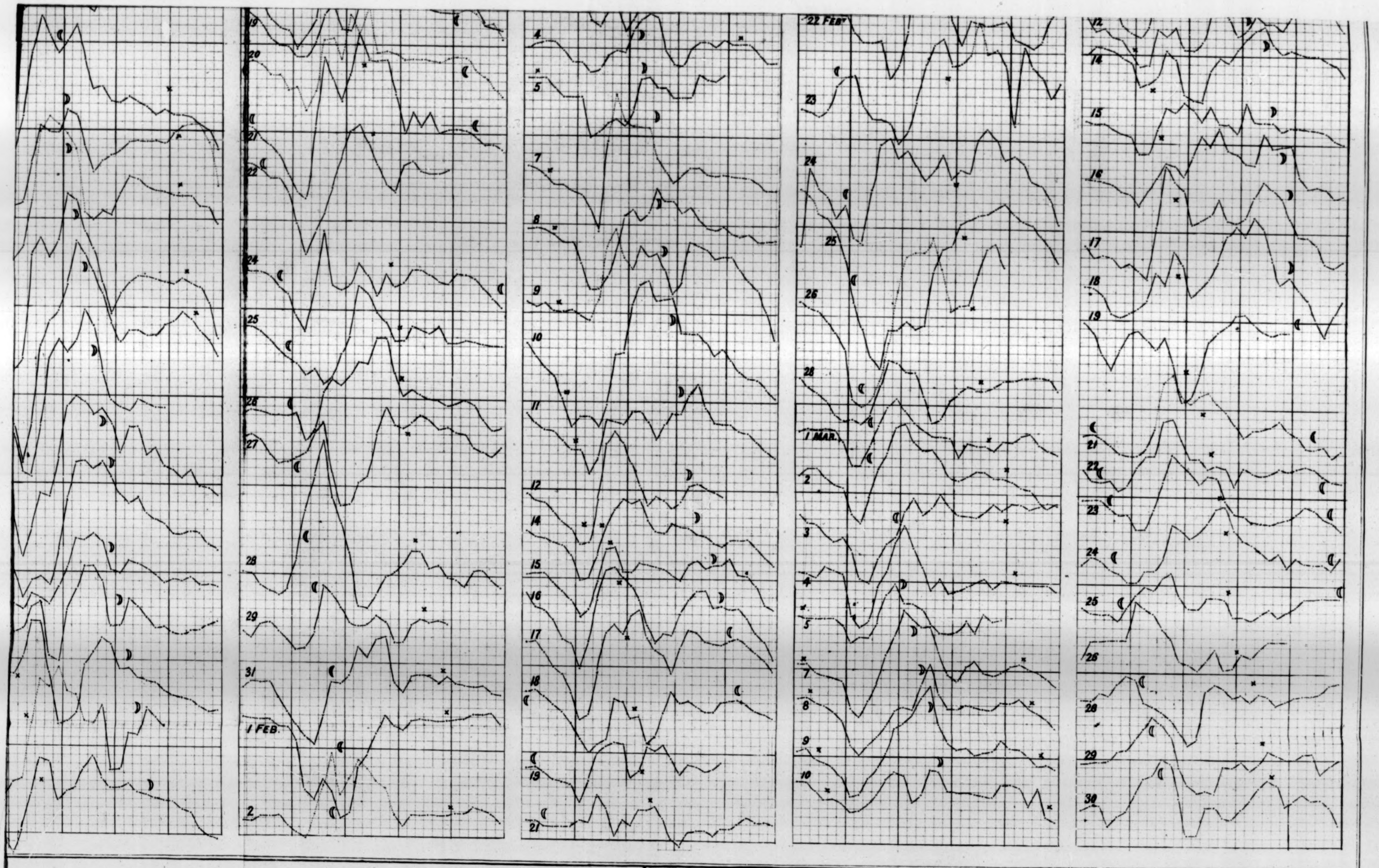
DAILY CURVES OF MAGNETIC DECLINATION AT TREVANDRUM, FROM 18TH DECEMBER, 1858, TILL 30TH MARCH, 1859. BY J. A. BROUN, F.R.S.

PLATE XLII.



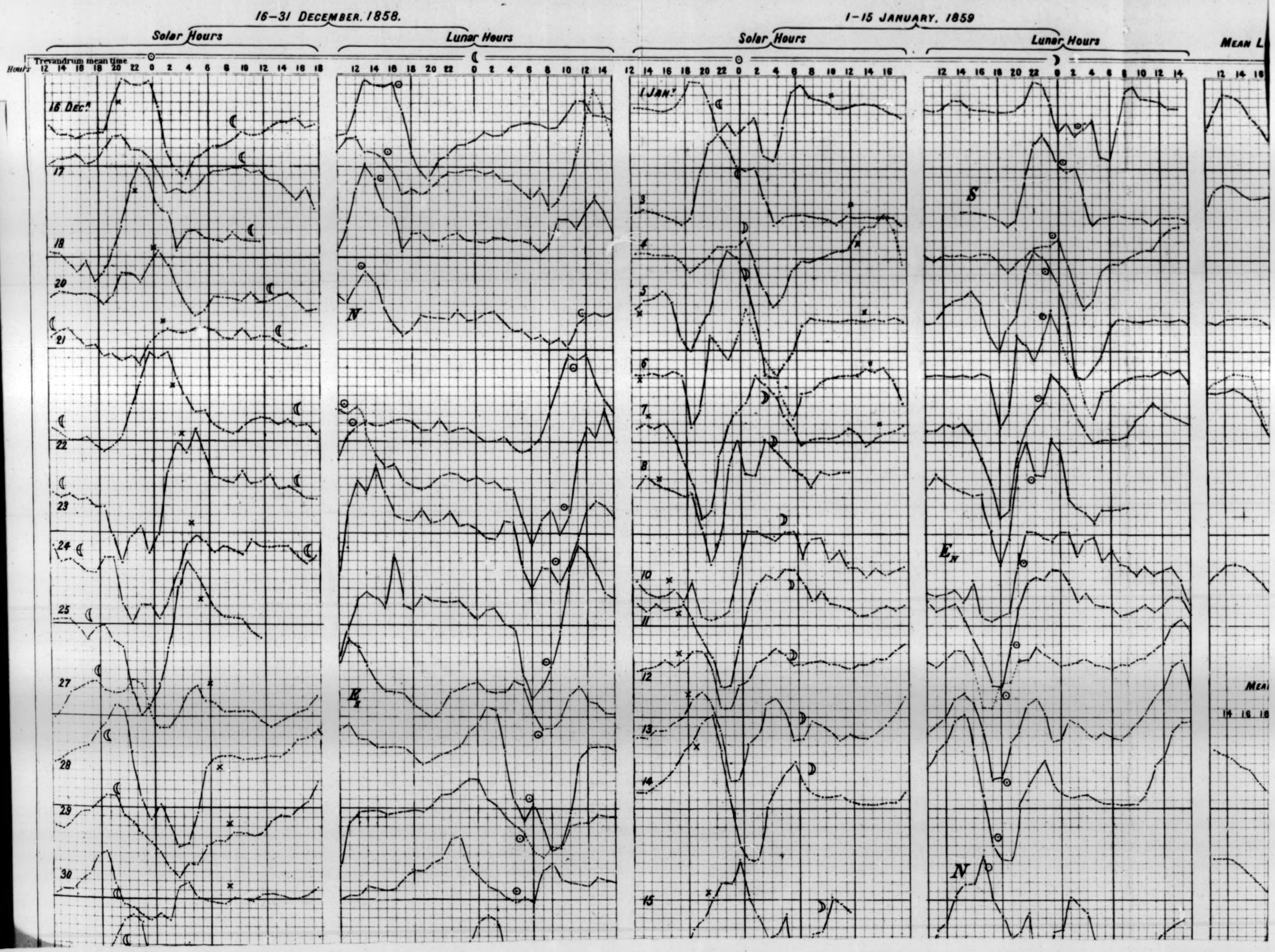


(( )) Moon on the superior meridian. × Moon on the inferior meridian. One division = 0'.25 ; One inch = 2'.5. Ascending Curves indicate movement of the North end of the n



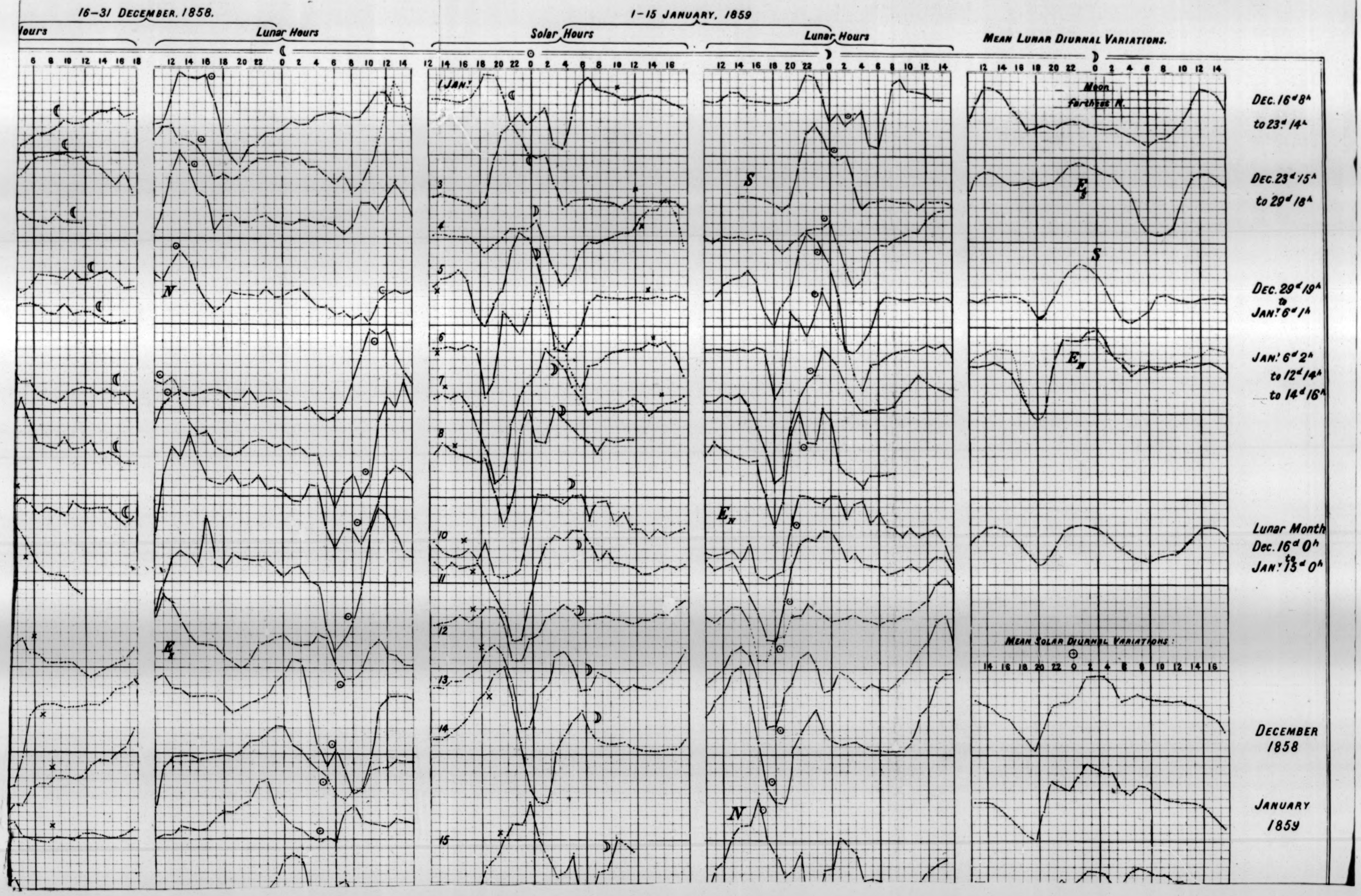
erior meridian. × Moon on the inferior meridian. One division = 0'.25; One inch = 2'.5. Ascending Curves indicate movement of the North end of the needle towards the East.

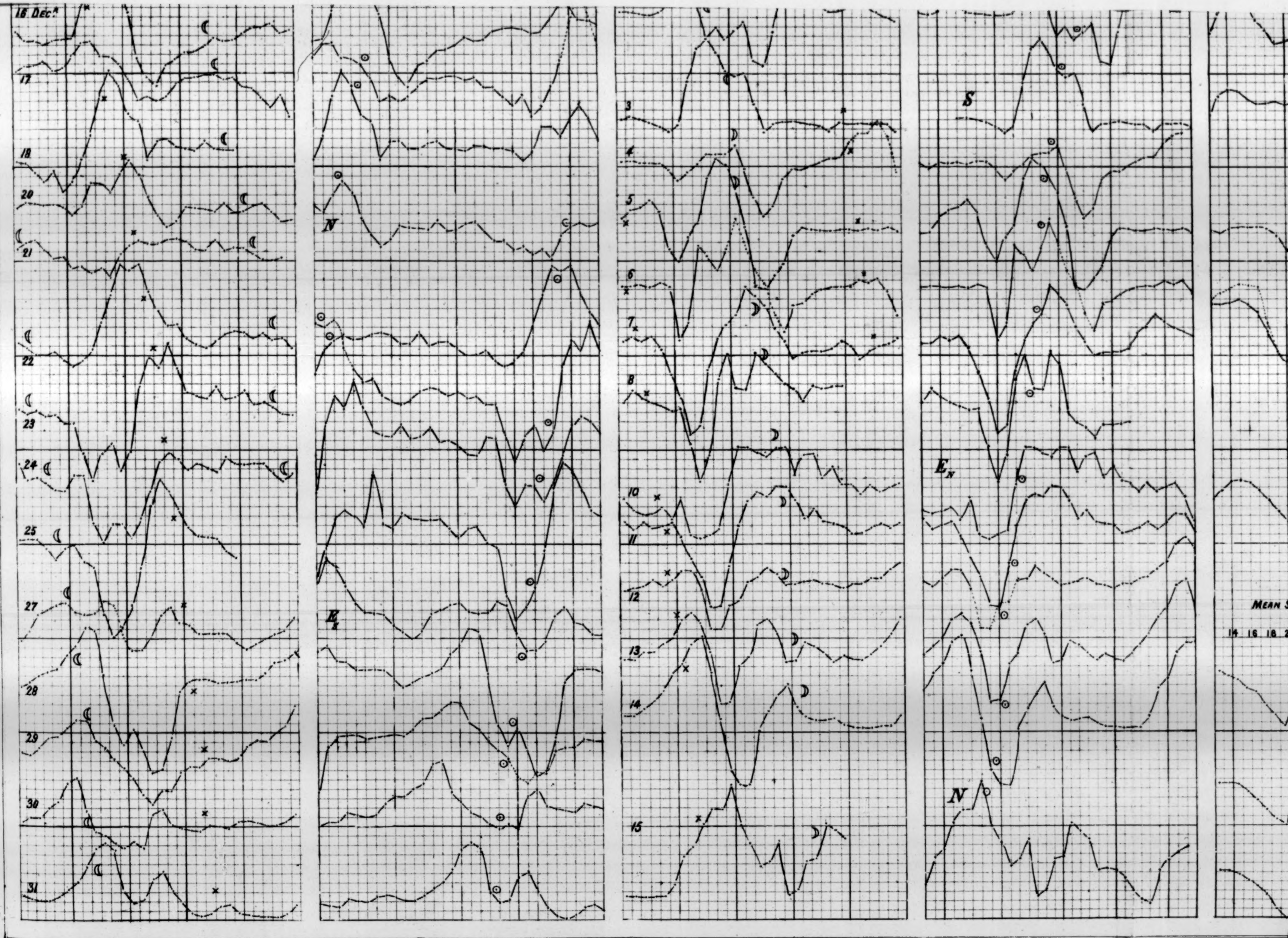
Curves of Differences of Daily Movements from the Mean Solar Diurnal Variation, and Resulting Lunar Diurnal Variations, at Trevandrum. Deduced by J. A. Brou



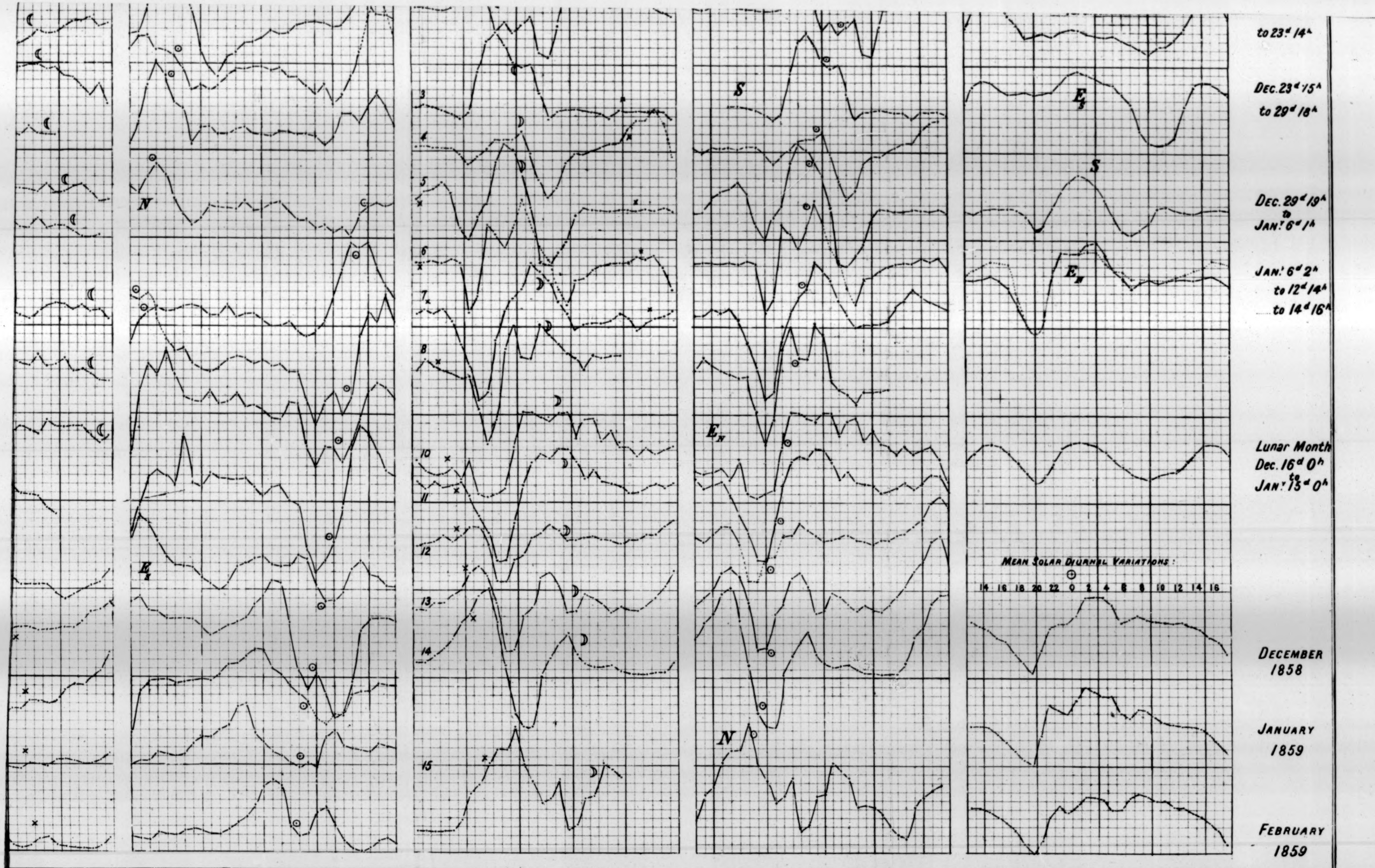
CES OF DAILY MOVEMENTS FROM THE MEAN SOLAR DIURNAL VARIATION, AND RESULTING LUNAR DIURNAL VARIATIONS, AT TREVANDRUM, DEDUCED BY J. A. BROUN, F.R.S.

PLATE XLIII.





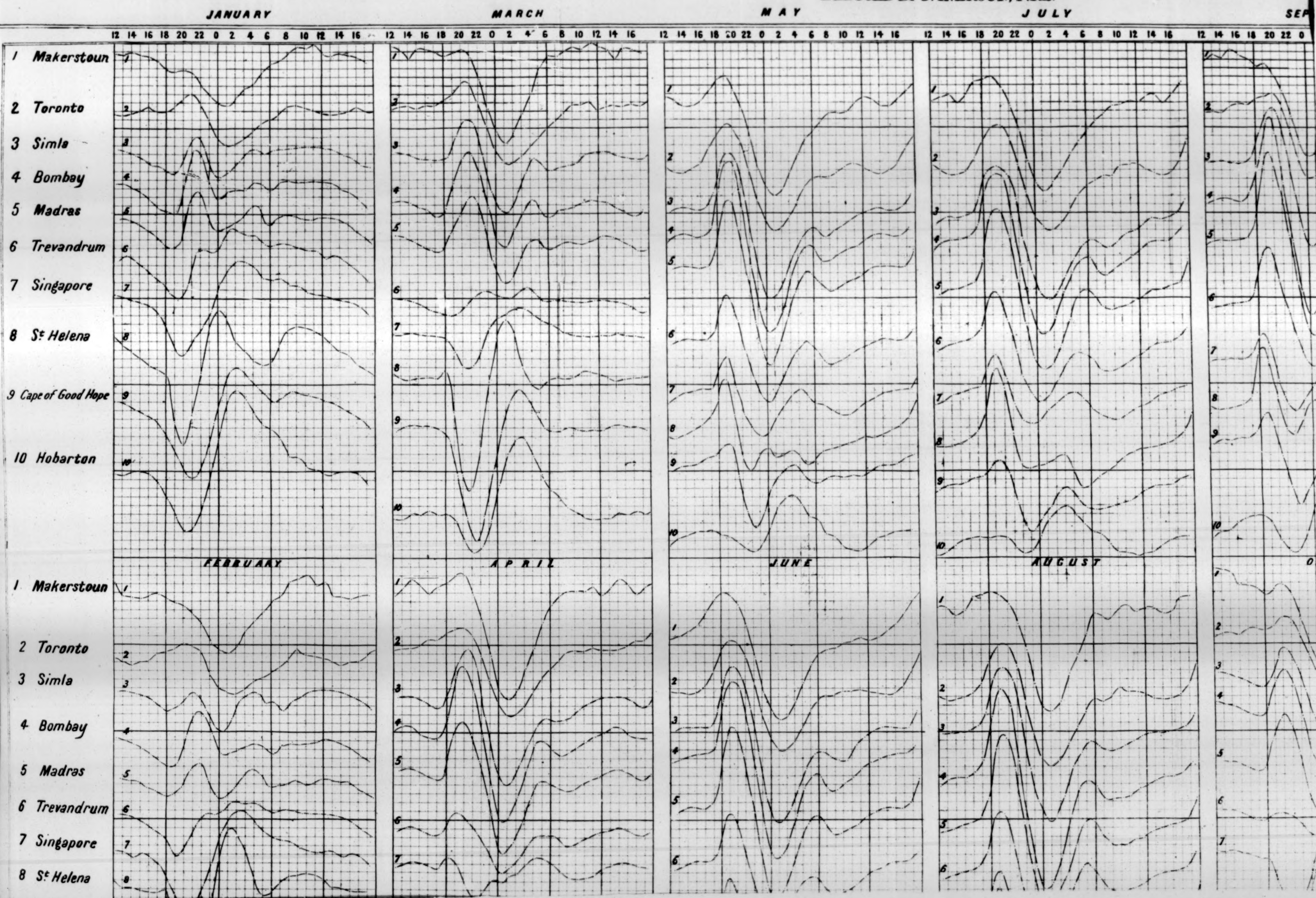
() Moon on the superior meridian. × Moon on the inferior meridian. ⊙ Sun on the meridian. One division = 0'.25. One inch = 2'.5. Ascending curves indicate movement of the North End



rior meridian. × Moon on the inferior meridian. ⊙ Sun on the meridian. One division = 0'.25. One inch = 2'.5. Ascending curves indicate movement of the North End of the needle towards the East.

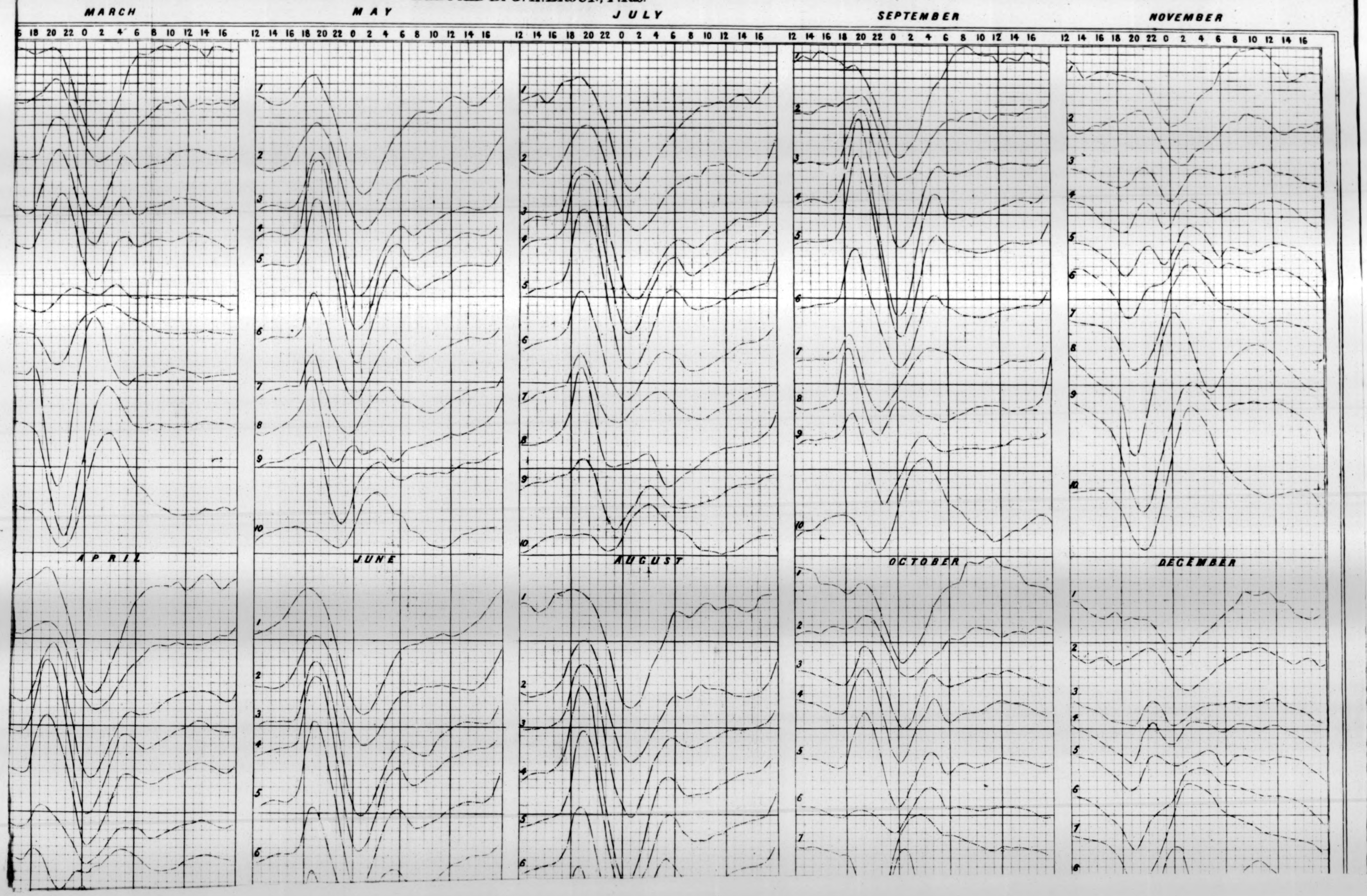
Curves of the Diurnal Variations of Magnetic Declination, as shown by a Needle freely suspended in the direction of the Magnetic Inclination, at Ten Stations betwi

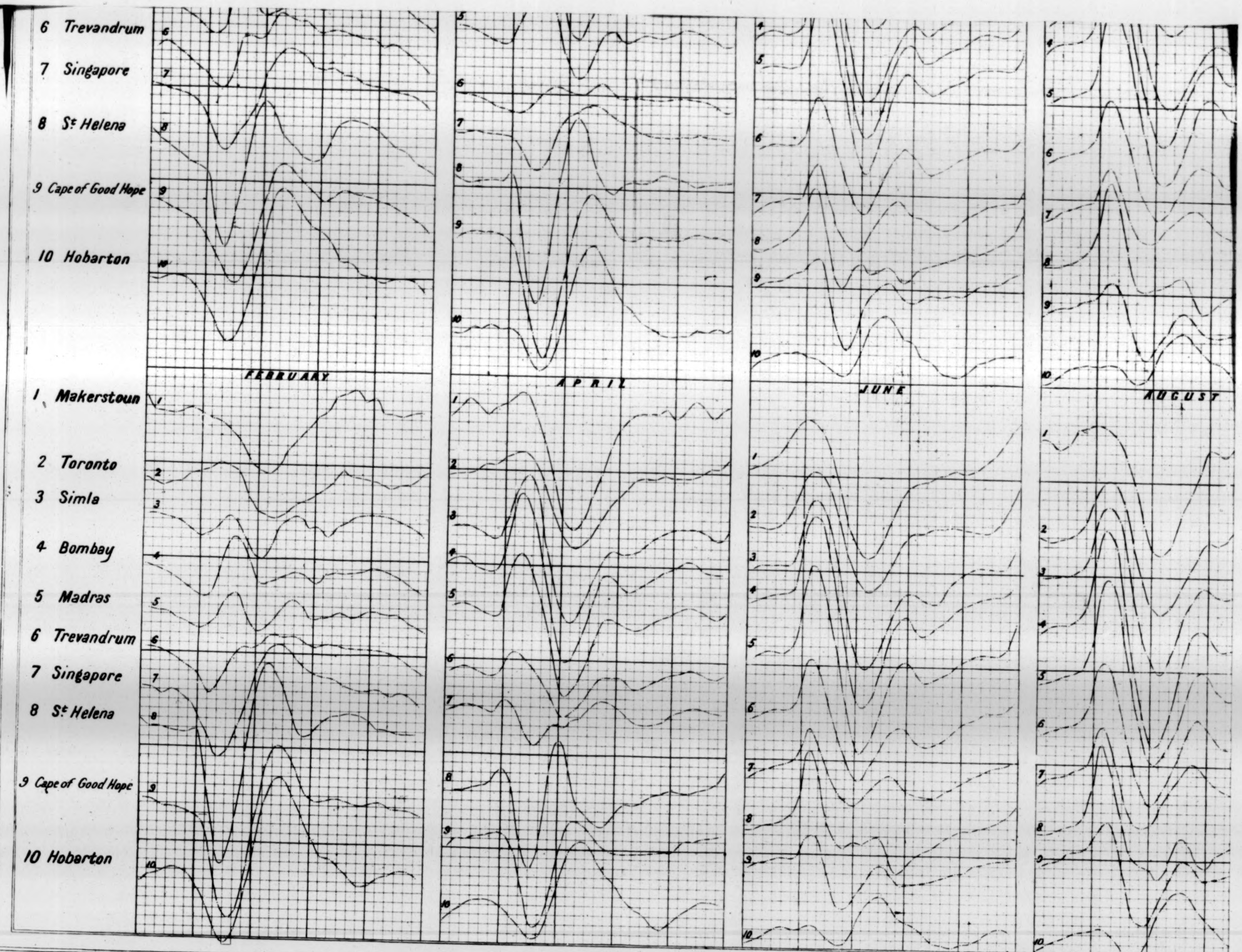
Deduced by J. A. Broun, F.R.S.



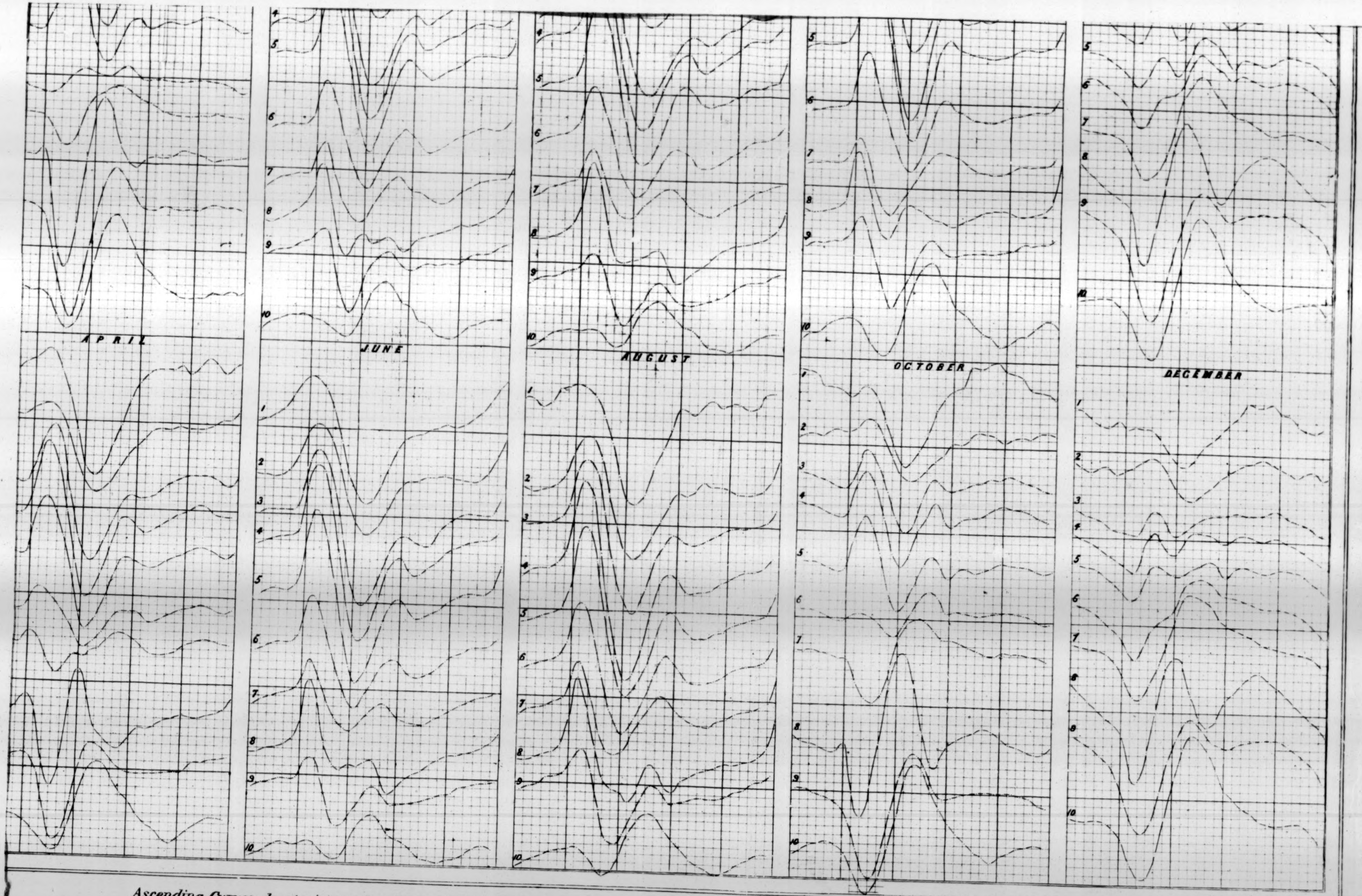
MAGNETIC DECLINATION, AS SHOWN BY A NEEDLE FREELY SUSPENDED IN THE DIRECTION OF THE MAGNETIC INCLINATION, AT TEN STATIONS BETWIXT LATITUDES 56 N. & 43 S., FOR EACH MONTH OF THE YEAR.

DEDUCED BY J. A. BROUN, F.R.S.





Ascending Curves denote increasing Easterly Declination. One Division = 0'.25;



Ascending Curves denote increasing Easterly Declination. One Division = 0′.25; One Inch = One minute.

XLVI.—*On an Application of Mathematics to Chemistry.* By Alexander Crum Brown, M.D., D.Sc.

(Read 18th February 1867.)

Prefatory Note, added June 24, 1867.

Since reading this paper, I have seen Sir Benjamin Brodie's paper on "The Calculus of Chemical Operations," read before the Royal Society of London.

The two papers resemble one another, merely in being applications of mathematical language to chemistry. They entirely differ in method, object, and result.

I may here mention, what I have omitted to state explicitly in the paper, that I have no idea of attempting to substitute a functional notation for that in common use. I only propose to use a functional notation to express certain general and serial relations in those cases where the common atomic notation is inconvenient or obscure.

The full application of Mathematics to Chemistry can only be made when a fundamental physical hypothesis is discovered, from which, by means of mathematical methods, results may be deduced which coincide with the observed facts of chemistry. In the meantime, however, a profitable application of mathematics may be made in another direction. Mathematics is the science by means of which consequences are deduced from laws; and although we have not, as yet, discovered the laws of chemistry, we have what may represent them,—approximate generalisations. To these we can apply mathematical methods, or at least we can express them in mathematical language.

I.—*Definition of Symbols.*

The objects of chemical study are of two classes—1*st*, substances; and 2*d*, processes performed on these substances. In this paper I shall represent the former as operands, and the latter as operators; where the contrary is not specially mentioned, a single operand symbol will be used to indicate one molecule, or chemical unit of a homogeneous substance.

An operator ϕ is defined by the chemical equation or equations connecting a, and $\phi \cdot a$, a being a molecule of a substance, and $\phi \cdot a$ the result of applying the process ϕ to it. Thus if the chemical equation be $a + KHO = \phi \cdot a + H_2O$, the process ϕ as applied to a, is the union of a with a molecule of caustic potash and the simultaneous elimination of a molecule of water.

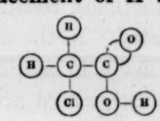
II.—*Combination of Symbols.*

The sign + will be used in a purely enumerative sense, to connect operands. Thus, $a + b + c +$ &c., means one molecule of a, and one of b, and one of c, &c. In the same way na means n molecules of a, the sign + and numerical multipliers never being used to indicate combination.

Many chemical processes are capable of being applied either to *one* molecule or to several simultaneously. Thus, if ϕ be defined by the equation $a + NH_3 = \phi \cdot a + 2HCl$, or by a series of equations equivalent to this, ϕ may be performed either on one molecule containing two atoms of chlorine, or on two molecules, each containing one. In the latter case, a becomes $a + b$, and $\phi \cdot a$ becomes $\phi \cdot (a + b)$, here ϕ is not performed on a *and* on b, but on $(a + b)$; that is, partly on a and partly on b, so that $\phi \cdot (a + b)$ is not equal to $\phi \cdot a + \phi \cdot b$. The distributive law does not, therefore, hold good here. $(a + b)$ is, however, the same as $(b + a)$, and generally we may change the order of operands connected by the sign +.

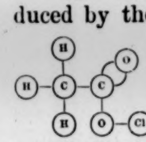
In $\phi \cdot (a + b + c +$ &c.), when a, b, c, &c. are identical, we have $\phi \cdot na$, and just as $\phi \cdot (a + b + c +$ &c.) is not equal to $\phi \cdot a + \phi \cdot b + \phi \cdot c +$ &c., so $\phi \cdot na$ is not equal to $n \phi \cdot a$; in other words, the commutative law of multiplication does not hold good for operators and numerical multipliers.

If a new process ψ be performed on the operand $\phi \cdot a$, there are two ways in which it may act; 1*st*, the process ψ may act *independently* on a, thus if $a = CH_4$ and ϕ be the replacement of H by COHO, and ψ the replacement of H by Cl, ϕ and ψ may act independently, giving chloracetic acid,



$\psi \cdot (\phi \cdot a)$

; or the new operator ψ may act on that part of the molecule which has been introduced by the process ϕ, giving in the instance above the acetate of chlorine,



$\psi \cdot (\phi \cdot a)$

$(\psi \cdot \phi \cdot a)$ may therefore have two quite distinct meanings, and it will be advisable to have two forms of notation for it. I shall represent the first by $\left.\begin{matrix}\phi\\ \psi\end{matrix}\right| \cdot a$, and the second by $\psi \cdot \phi \cdot a$, the complex operator $\psi \cdot \phi$ being in the case cited above, the replacement of H by COHO, in which the H is replaced by Cl, and may, therefore, properly enough be said to be the operator ϕ, acted on by ψ.

In the first case, which may be called the *vertical multiplication of operators*, I

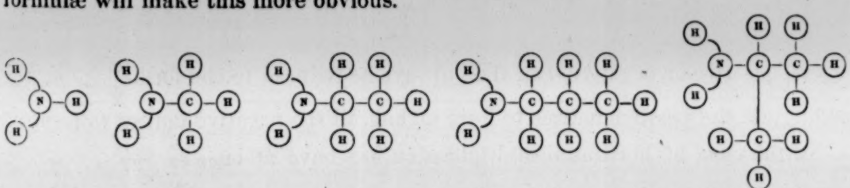
shall assume for the present, that as the operators act independently, their order may be varied without change of meaning, and that the commutative law of multiplication applies.* In the second case, which may be called the *horizontal multiplication of operators*, it is obvious that the order cannot be changed without changing the meaning,—here then the commutative law does not apply, $\phi\cdot\psi\cdot a$ being in general different from $\psi\cdot\phi\cdot a$.

III.—The Laws of positive Integral Indices, as applied to Operators.

In the case of vertical multiplication, we may represent $\left.\begin{array}{c}\phi\\ \phi\end{array}\right|\cdot a$ by the symbol $\phi^{2]}\cdot a$, $\left.\begin{array}{c}\phi\\ \phi\\ \phi\end{array}\right|\cdot a$, by $\phi^{3]}\cdot a$, &c., and generally, it is obvious, that $\left.\begin{array}{c}\phi^{m]}\\ \phi^{n]}\end{array}\right|\cdot a = \phi^{m+n]}\cdot a$ and $\left(\phi^{m]}\right)^{n]}\cdot a = \phi^{mn]}\cdot a$, when m and n are positive integers.

In horizontal multiplication, $\phi\cdot\phi\cdot a$, may be written $\phi^2\cdot a$, $\phi\cdot\phi\cdot\phi\cdot a$, $\phi^3\cdot a$, &c., understanding that $\psi\cdot\phi^n\cdot a$ is merely a contraction for $\psi\cdot\phi\cdot\phi\cdots\phi\cdot a$, and also, that $\phi^n\cdot\psi\cdot a$ is $\phi\cdot\phi\cdots\phi\cdot\psi\cdot a$; in other words, that the complex operator ϕ^n always acts by means of the ϕ at the extreme right of the series, and that an operator acting on it always acts upon the ϕ at the extreme left. With these assumptions, we at once see that the laws of indices $\phi^m\cdot\phi^n\cdot a = \phi^{m+n}\cdot a$, and $\left(\phi^m\right)^n\cdot a = \phi^{mn}\cdot a$, hold good also in the case of horizontal multiplication when m and n are positive integers.

It also follows from these assumptions, that if $\phi^n = \psi$, $\psi\cdot\phi = \phi^{n+1}$, but $\phi\cdot\psi$ may have several values, one of which is ϕ^{n+1}, for in $\phi\cdot\phi^n$, the action of ϕ is restricted to a particular part of ϕ^n, while no such restriction is made in the case of $\phi\cdot\psi$. Thus, if a be ammonia, and ϕ the replacement of H by CH_3, $\phi^2\cdot a$ is ethylamine, and $\phi\cdot\phi^2\cdot a = \phi^3\cdot a$ is propylamine; but if we put $\phi^2 = \psi$, $\phi\cdot\psi\cdot a$ may be either propylamine ($\phi^3\cdot a$) or isopropylamine ($\phi^{2]}\cdot\phi\cdot a$). The graphic formulæ will make this more obvious.



a $\phi\cdot a$ $\phi^2\cdot a$ $\phi\cdot\phi^2\cdot a = \phi^3\cdot a = \phi\cdot\psi\cdot a$ $\phi^{2]}\cdot\phi\cdot a = \phi\cdot\psi\cdot a$

* It is not by any means certain that this is true, even in the case of the simplest operands; it is almost certain that it is not true in the case of complex operands; but as we have not sufficient data to enable us to form a theory connecting the order in which operators are applied to a molecule with the parts of the molecule upon which they act, I have provisionally assumed the simplest possible law.

ψ being here the replacement of H by C_2H_5, $\phi\cdot\psi$ is the replacement of H by C_2H_5, in which H is replaced by CH_3; ϕ^2 on the other hand, is the replacement of H by CH_3 in which H is replaced by CH_3 and $\phi\cdot\phi^2$, is the replacement of H by CH_3 in which H is replaced by CH_3 in which H is replaced by CH_3.

IV.—*On Negative Integral Indices as applied to Operators.*

Without contradicting any previous assumption, we may define the symbol ϕ^{-1} by the equation $\phi\cdot\phi^{-1}\cdot a = a$, or $\phi\cdot\phi^{-1} = 1$. It is at once obvious that $\phi\cdot\phi^{-1} = \phi^{-1}\cdot\phi$, for if ϕ be the replacement of A by B, ϕ^{-1} is the replacement of B by A, and $\phi\cdot\phi^{-1}$ is the replacement of B by A, in which A has been replaced by B, that is, the replacement of B by B, and similarly $\phi^{-1}\cdot\phi$ is the replacement of A by A; it is different, however, in the case of vertical multiplication, $\left.\begin{matrix}\phi^{-1}\\ \phi\end{matrix}\right|\cdot a$ is not necessarily equal to a, for $\left.\begin{matrix}\phi^{-1}\\ \phi\end{matrix}\right|\cdot a$ expresses the replacement of A by B in one part of the molecule, and of B by A in another part of it, and it is only in particular cases that these two processes will leave the molecule unchanged, $\left.\begin{matrix}\phi^{-1}\\ \phi\end{matrix}\right|\cdot a$ must, of course, be *isomeric* with a.

$\left.\begin{matrix}\phi^{-1}\\ \phi^{-1}\end{matrix}\right|\cdot a$ is from the last chapter $\left(\phi^{-1}\right)^{2]}\cdot a$, and no confusion will arise by writing this $\phi^{-2]}\cdot a$, as the only other meaning which this symbol could have would be $\left(\phi^{2]}\right)^{-1}\cdot a$, and this would be defined by the equation $\phi^{2]}\cdot\left(\phi^{2]}\right)^{-1}\cdot a = a$, which could only mean $\left.\begin{matrix}\phi\cdot\phi^{-1}\\ \phi\cdot\phi^{-1}\end{matrix}\right|\cdot a$, giving as the equivalent for $\left(\phi^{2]}\right)^{-1}$, the expression $\left(\phi^{-1}\right)^{2]}$. We have thus, generally, $\left.\begin{matrix}\phi^{-m]}\\ \phi^{-n]}\end{matrix}\right|\cdot a = \phi^{-(m+n)]}\cdot a$, and $\left(\phi^{-m]}\right)^{n]}\cdot a = \phi^{-mn]}\cdot a$. An expression, $\left.\begin{matrix}\phi^{p]}\\ \phi^{q]}\\ \phi^{r]}\\ \text{\&c.}\end{matrix}\right|\cdot a$ (where p, q, r, &c. are any integers, positive or negative) can then always be reduced to the form $\left.\begin{matrix}\phi^{m]}\\ \phi^{-n]}\end{matrix}\right|\cdot a$, by adding all the positive indices to form m, and all the negative indices to form n.

In the case of horizontal multiplication, we have at once $(\alpha\cdot\beta\cdot\gamma\ \ldots\ldots\ \nu)^{-1}\cdot a = \nu^{-1}\ \ldots\ldots\ \gamma^{-1}\cdot\beta^{-1}\cdot\alpha^{-1}\cdot a$, and therefore $\left(\phi^{n}\right)^{-1} = \left(\phi^{-1}\right)^{n}$, and we may, therefore, write this ϕ^{-n}; and as $\phi\cdot\phi^{-1}\cdot a = a$, we have $\phi^2\cdot\phi^{-2}\cdot a = \phi^{-2}\cdot\phi^2\cdot a = a$; and, generally, $\phi^m\cdot\phi^{-n}\cdot a = \phi^{m-n}\cdot a$; also $\left(\phi^{-m}\right)^{n}\cdot a = \phi^{-mn}\cdot a$ and

$\left(\phi^m\right)^{-n}\cdot a = \phi^{-mn}\cdot a$; or, the ordinary laws of indices hold good in horizontal multiplication where m and n are any integers, positive or negative.*

V.—*Fractional Indices as applied to Operators.*

The symbols $\phi^{\frac{1}{n}}$ and $\phi^{\frac{1}{n}]}$ may be defined thus $\left(\phi^{\frac{1}{n}}\right)^{n}\cdot a = a$ and $\left(\phi^{\frac{1}{n}]}\right)^{n]}\cdot a = a$. When ϕ is of the form ψ^n, the first expression will have a real meaning, and so will the second when ϕ is of the form $\psi^{n]}$. In the opposite case, $\phi^{\frac{1}{n}}$ and $\phi^{\frac{1}{n}]}$ are purely imaginary operators, and it is only when n of them are multiplied together in the first case horizontally, and in the second vertically, that they acquire a meaning. It may, however, be useful to break up an operator into imaginary factors. Thus, if ϕ be the replacement of three atoms of hydrogen by one of nitrogen, $\phi^{\frac{1}{3}]}$ is the replacement of H by $\frac{1}{3}$ N, and where the 3 H atoms replaced by N have different positions in a molecule, it may be convenient to express it in this way. Thus, if a be benzol, ψ the replacement of H by NH_2, and ϕ the replacement of 3 H by N, we may express diazobenzol by $\left.\begin{array}{c}\phi^{\frac{1}{3}]}\cdot\psi \\ \phi^{\frac{1}{3}]}\end{array}\right|\cdot a$; thus indicating that the process ϕ is performed on $\psi\cdot a$, so that $\phi^{\frac{1}{3}]}$ acts on ψ, and $\phi^{\frac{1}{3}]}$ on a.

In the same way, an operator may be broken up into imaginary factors which are not identical; thus, if ϕ be the replacement of N by O″ and HO, we may put $\phi\cdot a = \left.\begin{array}{c}\alpha \\ \beta\end{array}\right|\cdot a$ where α is the replacement of $\frac{2}{3}$N by O″, and β the replacement of $\frac{1}{3}$N by HO.

VI.—*On Chemical Groups and Series.*

Those groups which in a former paper (*Trans.* xxiv. 331) I have called chemical genera, and which are characterised by containing a particular radical, the *generic* radical, may each be represented by a generic formula $\phi\cdot X$, where ϕ is an operator introducing the generic radical, and X a molecule or group of

* There is an obvious and important difference between direct and inverse operators, which I may mention here.

If we represent as direct those operators only which express direct processes which can be performed so as to get $\phi\cdot a$ by acting on a, it is plain that if ϕ includes the physical conditions of the process, $\phi\cdot a$ can have only *one* value. It may have a number of conceivable values, and it is the business of the experimental chemist to find out which of these is the real one; but $\phi^{-1}a$ may have more than one *real* value, for ϕ^{-1} is not restricted to mean *one* actual process capable of being performed, but any process, such that $\phi^{-1}a$ shall be by the process ϕ (performed on the part of the molecule, introduced or modified by ϕ^{-1}) reproduce a. Thus, if ϕ be the addition of H_2, and a be alcohol, $\phi^{-1}a$ may represent either aldehyde or oxide of ethylene, for both of these give a when treated by the process ϕ. Examples of this kind might be multiplied to any extent.

molecules on which ϕ can act. Thus, if ϕ be the replacement of H by COHO, the monocarbon acids, the dicarbon acids, and the tricarbon acids, have the formulæ $\phi \cdot X$, $\phi^{2]} \cdot X$, $\phi^{3]} \cdot X$, respectively.

I shall only consider further the application of this method to two other kinds of series, which we may call *functional chemical series.* The first consists of terms of the form $a, \phi \cdot a, \phi^{2]} \cdot a, \ldots \phi^{n]} \cdot a$. Here the successive terms are produced by the repetition of the process ϕ *independently* on a, and we may call such series *independent functional series.* We have instances of this in the series CH_4, CH_3Cl, CH_2Cl_2, $CHCl_3$, CCl_4; NH_4I, NH_3CH_3I $NH_2(CH_3)_2I$, $NH(CH_3)_3I$, $N(CH_3)_4I$; C_2H_2, C_2H_4, C_2H_6, or in the so-called homologues of benzol C_6H_6, $C_6H_5CH_3$, $C_6H_4(CH_3)_2$, &c. In all these series we have a common difference between successive terms, and we may observe that all such series necessarily consist of a *definite* number of terms, for the process ϕ can only be performed *independently* on a a definite number of times. If we put $a = \phi^{-n]} \cdot b$, where b is not of the form $\phi^{-1} \cdot x$, n is the number of times ϕ can be performed independently on a, and $n + 1$ is the number of terms in the series.

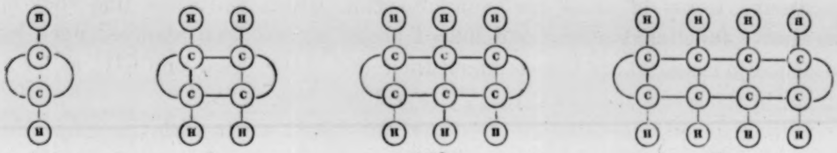
The other form of functional series is $a, \phi \cdot a, \phi^2 \cdot a, \ldots \phi^n \cdot a$. In this the successive terms are derived by the repetition of ϕ on that part of the molecule which was introduced or modified by the previous performance of ϕ, and we may call such a series a successive functional series. In order that such a series may be possible, it is necessary that a be of the form $\phi^{-1} \cdot b$, and that ϕ be of the form $\phi^{-1} \cdot \psi$; in other words, that ϕ can be applied once to a and once to ϕ. We have examples of this kind of series in "homologous" series, such as NH_4I, $NH_3(CH_3)I$, $NH_3(CH_2\overline{CH_3})I$, &c.

Here the law of derivation of successive terms does not, as in the independent functional series, contain in itself a determination of the number of terms of which the series consists, except where ϕ diminishes the weight of the molecule on which it acts. In this case, however, by inverting the series, we find that there are an indefinite number of terms before a, so that by inversion, if necessary, all such series may be reduced to the form $a, \phi \cdot a \cdots \phi^n \cdot a$, where a is not of the form $\phi \cdot x$, and is, therefore, the first term.

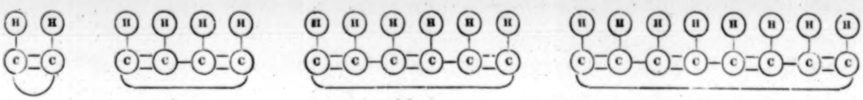
In a series of this form, if ϕ does not involve x in the expression $\phi \cdot x$, we have a common difference in weight and composition. This is the case in "homologous" series. If ϕ involve x, there cannot, of course, be a common difference; this case will, however, be examined in a subsequent part of this paper.

There are two varieties of the successive functional series—1st, Where ϕ is a simple addition; and, 2d, Where ϕ is a replacement. In the first, the radical (or radicals) necessarily artiad, must be such that it (or they) can be again added to that part of the molecule which has been introduced or modified by the first ϕ. We have an example of this kind of series in the polymers of acetylene,

where ϕ consists in the addition of the radical C_2H_2  This series may be represented in several different ways, and we do not as yet know which is the true one. I append graphic formulæ of two of these ways, to illustrate my meaning—

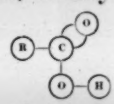


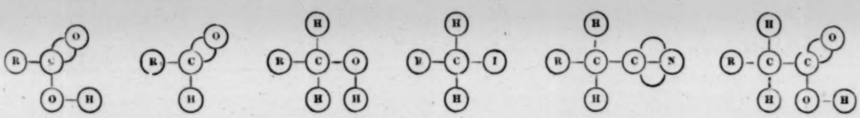
$a \qquad \phi \cdot a \qquad \phi^2 \cdot a \qquad \phi^3 \cdot a$



$a \qquad \psi \cdot a \qquad \psi^2 \cdot a \qquad \psi^3 \cdot a$

The second of these is the form proposed by KEKULÉ, and appears on the whole to be the most probable.

In the other variety of successive functional series, ϕ is the replacement of one or more radicals by one or more new radicals. Let the replaced radical or group of radicals, be represented by Γ, and the replacing radical or group of radicals by Λ, and we at once see, first, that Γ and Λ are equivalent; and, second, that Λ must contain Γ, as the process is capable of repetition. When ϕ in $\phi \cdot x$ is independent of x, the difference between Λ and Γ is constant (as indeed Λ and Γ are themselves constant throughout the series), and is the common difference of the series. This is the case in "homologous" series. As an example, we may take the series of the fatty acids. The formula of one of them is  where R is of the form C_nH_{2n+1}; and taking any of the series of substances through which it is connected with the next higher member of the series, such as



we see that Γ is O'' and $(HO)'$, and Λ is H', H', and $(COHO)'$. Λ thus contains Γ, and the process is capable of repetition. The common difference is here CH_2, but there are instances of series probably of the kind we are now considering (although we cannot at present trace the relation between successive terms) in which we have other common differences. Thus the carbonates, oxalates, and

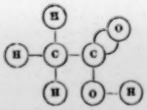
mesoxalates, form a series in which the common difference is CO. In the series —methylic alcohol, ethylenic glycol, glycerine, erythrite, the unknown body $C_5H_5(HO)_5$, and mannite, as also in the series tartronic acid, tartaric acid, the unknown trioxypyrotartaric acid, and mucic acid, we have a common difference of CH(HO). It is quite possible that relations may be discovered between the successive terms of these and similar series, which will show that they are successive functional series; and indeed homology has been assumed upon less evidence in cases where the common difference is the usual one CH_2, as for instance in the series of bases $C_nH_{2n-5}N$ (pyridine, picoline, &c.).

It is instructive to compare *successive* functional series with the *independent* functional series isomeric with them. Thus, ammonia, methylamine, dimethylamine, trimethylamine, with ammonia, methylamine, ethylamine, propylamine: or benzol, toluol, xylol, pseudocumol, &c., with benzol, toluol, ethylbenzol, propylbenzol, &c. KEKULÉ, to whom we are greatly indebted for clearness of conception of the structure of the aromatic bodies, calls both of the last-mentioned series "homologous;" as the word "homologous" has never, as far as I know, been strictly defined, there can be no objection to this; but if we call the series C_6H_6, C_6H_5 (CH_3), $C_6H_4(CH_3)_2$, &c. homologous, we must apply the same term to the series ammonia, methylamine, dimethylamine, &c. It will be observed, that while no *three* successive terms of an *independent* functional series can belong to the same *successive* functional series, *any* two successive terms may. For $\phi^{n]}\cdot a$ and $\phi^{n+1]}\cdot a$ are $\phi^{n]}\cdot a$, and $\left.\begin{matrix}\phi^{n]}\\ \phi\end{matrix}\right|\cdot a$, and these are the initial terms of the series $\left.\phi^{n]}\right|\cdot a$, $\left|\begin{matrix}\phi^{n]}\\ \phi\end{matrix}\right|\cdot a$. $\left.\begin{matrix}\phi^{n]}\\ \phi^2\end{matrix}\right|\cdot a \cdots \left.\begin{matrix}\phi^{n]}\\ \phi^n\end{matrix}\right|\cdot a$. Thus, methylamine, dimethylamine, and trimethylamine, are not in successive series, but methylamine and dimethylamine are the initial members of the series which continues methyl ethylamine, methyl propylamine, &c.

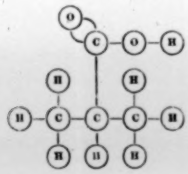
If we restrict the term "homologous series" to successive functional series having the common difference CH_2, we see that one subtance may be the starting point of several homologous series. In this sense, NOAD's toluic acid and a toluic acid are both homologues of benzoic acid. The one series being, benzoic acid, methyl benzoic acid (NOAD's toluic acid), ethyl benzoic acid, propyl benzoic acid, &c.; the other being benzoic acid, a toluic acid, the acid $C_6H_5\cdot CH_2\cdot CH_2$ $\cdot$COHO, &c.

When ϕ in the expression $\phi\cdot x$ involves x, it is obvious that there can be no common difference in the series a, $\phi\cdot a$, $\phi^2\cdot a \cdots \phi^n\cdot a$.

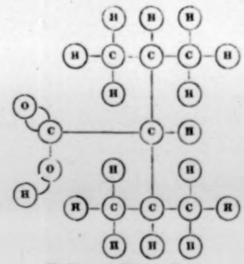
Such series have not been specially examined, but examples can easily be suggested; for instance, let ϕ be the series of processes by which acetic acid is successively converted into acetone, acetonic acid, and isobutyric acid; then ϕ can again be applied to the last mentioned acid, giving isobutyrone, isobutyronic acid, &c., thus giving rise to a series of acids, having the formulæ,—



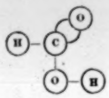
Acetic Acid.



Isobutyric Acid.



Unknown Third Term.

Formic acid  is formally the first term of the series, although as the formic ketone is unknown, we cannot directly prove the relation.

It is, however, unnecessary to dwell longer on such series; it is sufficient to point out the probability of their existence, in order to show that a common difference is not an essential character of successive functional series.

XLVII.—*Description of* Pygopterus Greenockii (Agassiz), *with Notes on the Structural Relations of the Genera* Pygopterus, Amblypterus, *and* Eurynotus. By Ramsay H. Traquair, M.D., Demonstrator of Anatomy in the University of Edinburgh. Communicated by William Turner, M.B. (Plate XLV.)

(Read 5th March 1866.)

Professor Agassiz, in his " Poissons Fossiles," vol. ii. p. 78, has mentioned a species of *Pygopterus* as occurring in the carboniferous shales of Wardie, near Edinburgh, and which he has named *P. Greenockii,* in honour of Lord Greenock, the first collector of ichthyolites from that locality. He has, however, neither given a figure of this species nor any description of it, beyond saying that the known fragments consist of hardly anything but heads, with the anterior portion of the trunk, and that the scales covering this part of the body are higher than broad—a circumstance distinguishing them from the scales of all the other species of the genus.

Having for some time back collected fossils from this locality, I am enabled not only to figure a complete specimen of the fish, but also to describe its structure a little more in detail, though there are still many points concerning which more knowledge would be desirable.

These fishes occur in elongated nodules of clay ironstone, which are almost always divided by transverse fissures into a great many segments. When the nodule becomes detached from its shaly matrix by the action of the sea, these segments become separated and scattered; hence the difficulty of acquiring entire specimens. Heads are, however, more easily found, as the anterior extremity of the fish is usually imbedded in a larger and more compact piece of ironstone than those into which the rest of the nodule divides. To obtain an entire specimen, it is then necessary to find the nodule *in situ*, to extract it from the shale piecemeal, and then carefully to split the separate pieces, which must, lastly, be properly assorted and glued together. The specimens are always much crushed, and often otherwise mutilated; the body of the fish presenting sometimes nothing more than a heap of disjointed scales. The texture of the ironstone is also unfavourable to the complete study of the individual parts, as it is almost impossible to develope or work out the specimen beyond what is exposed by the first splitting of the nodule.

It is also very difficult to obtain a good view of the external sculpturing of the scales and of the bones of the head, as these generally adhere with their

outer surface to the matrix; often, indeed, the scales themselves are so split, that neither outer nor inner surface, but a section, is visible.

The specimen shown in fig. 1 equals $14\frac{1}{4}$ inches in length, and $2\frac{5}{8}$ inches in breadth opposite the ventral fins. The other measurements are as follows:—

From tip of snout to opposite the origin	of dorsal fin,	. .	8 inches 8 lines.
" "	of ventral "	. .	6 inches.
" "	of anal "	. .	9 inches 1 line.
" "	of caudal "	. .	12 inches 4 lines.

Scales.—These are rather small in proportion to the size of the fish, but are much larger along the back and upper part of the sides than on the belly, where they are very minute. The larger scales are rather narrow, being, as AGASSIZ remarks, much higher than broad. The upper border of each scale projects in a very strong pointed spine, which is received into a corresponding excavation in the deep surface of the scale next above. Below the base of the spine the upper margin of the exposed portion of the scale is seen to be deeply concave, while the lower margin is rounded off into a blunt point, which fits on to the concave notch-like upper margin of the exposed portion of the scale next in order below. The whole of the exposed surface is ornamented by very delicate, nearly parallel, wavy ridges, whose general direction is from above downwards towards the lower angle of the scale, sometimes anastomosing with each other on the way.

The small scales along the body become more nearly equilateral, and those on the continuation of the vertebral column into the upper lobe of the tail-fin are also very small, and of an elongated lozenge shape. The largest scales are, however, those along the upper border of the tail. They are triangular in shape, acutely pointed behind, and notched in front, thus resembling very closely the corresponding scales in *Amblypterus* and *Eurynotus*, but being very much larger in proportion to the other scales. They are placed over each other in an imbricating manner, whereas very little imbrication is observable in the scales of the rest of the body,

Pectoral fins.—The specimen figured does not exhibit the pectoral fins, and I have seen no example of this fin complete to its termination. A fragment in my collection shows the pectoral to have been wide and powerful, the anterior rays strong, and with the transverse articulations farther apart than in the other fins, at least at their commencement. Another fragment shows that the anterior border of the pectoral fin was ornamented by a series of rather small imbricating fulcral scales.

The *ventral fin* of the right side is shown in fig. 1, V, its extremity being cut off, like those of the other fins of this specimen, by the edge of the nodule. Its base is wide, measuring in the present specimen one inch across, and the number of rays may be estimated at no less than 35. In the more posterior finer rays bifurcation may be observed, but the stronger anterior ones are cut off before

any such division takes place. The fulcral scales of the anterior margin are not seen, being obscured by the matrix.

The *dorsal fin* (fig. 1, D), measuring 1 inch 8 lines in breadth at its origin, is placed very far back, being nearly opposite the anal, and consists approximately of about 45 rays, which are rather coarse in the anterior two-thirds of the fin, but become finer posteriorly. The transverse articulations of the individual rays are rather close together, but the joints are still longer than broad, and the whole ray bifurcates a little beyond its middle. Fine, closely set fulcral scales are observed along the anterior margin of the fin.

The *anal fin* (fig. 1, A), like the dorsal, is large and strong; its base measures 1 inch 9 lines across. As in the latter, the transverse articulations of the rays are approximated; the joints of the first transverse row are, however, nearly twice as long as broad. The anterior margin is, as usual, set with small fulcral scales.

The *caudal fin* (fig. 1, C) is by far the largest, and, as far as its lower lobe is concerned, the longest of the azygos fins, the portion preserved of this lower lobe being, in the specimen figured, 1¾ inch long at its anterior margin; and we may safely suppose it to have been originally at least ¾ of an inch longer. It is, however, the worst preserved of all the fins of the specimen, so that all attempts to count its rays are in vain. These are somewhat finer than those of the dorsal and anal fins; their transverse articulations are much approximated, so that the joints are almost square. The stouter rays of the front of the lower lobe are not seen to bifurcate so far as they extend, till cut off by the edge of the nodule; the succeeding finer ones begin to split about their middle. The rays of the upper lobe are too badly preserved to admit of description, further than that they are comparatively short, and very fine. The tail, as a whole, is most typically heterocercal, the extremity of the body being continued along the upper lobe of the caudal fin, where it is defended by the two kinds of scales, lateral and mesial, already described.

On examining another specimen, not figured, I find that the rays of the caudal fin, like those of the anal of *Polypterus*, commenced below the skin in narrow flat pieces, imbricating from before backwards, and not divided by transverse articulations.

Internal skeleton.—I have in no instance found any trace of vertebral bodies, but, scattered among the scales of disjointed specimens, may often be seen pointed bony fragments, which must be the remains of the vertebral apophyses.

The *interspinous bones* of the azygos fins seem to have been well developed, as shown in a specimen not figured, but already referred to in the description of the rays of the caudal fin, and in which these bones, supporting the anal and caudal fin, are seen together, with traces of those belonging to the dorsal.

In connection with the anal fin, I counted at least 15 of these bones, which are very strong in front, the anterior one measuring 11 lines in length. They

become smaller posteriorly, but all are expanded at their extremities, these forming apparently a continuous line, to which the fin rays are articulated.

In the tail may be seen, in the same specimen, two sets of ossicles, an upper supporting the large azygos scales of the body prolongation, these being probably of the nature of neurapophyses· and a lower set supporting the rays of the caudal fin, and which may be regarded as the interspinous bones, or those in conjunction with the inferior spinous processes of the caudal vertebræ. These caudal interspinous bones are stout anteriorly, but become very small behind, where they assume a distinct hour-glass shape, being expanded at each extremity, and narrow in the middle, and where it may also be beautifully seen how each interspinous bone supports several fin rays. This must necessarily be the case in all the fins, seeing that the number of rays very considerably exceeds that of their supporting ossicles.

Head.—Figs. 2, 3, and 4 represent various views of the head of the specimen already alluded to in the description of the interspinous bones and of the caudal rays.

Fig. 2 is a profile view of the head, in which the eye is at once struck by the strength of the jaws, the extent of the gape, the very anterior position of the orbit, and the projection of the snout.

Fig. 3 is a view of the bones on the top of the cranium, and fig. 4 represents the snout seen from the front.

Referring to fig. 2, which in fact represents a *cast of the inner surface* of the cranial buckler, the bone having splintered off along with the matrix, leaving the sutures standing up as prominent lines, the lines radiating from the ossific centres being also well displayed, we find that some of the numerous bones exposed are easily enough recognised. Posteriorly, we have the parietals (7) very short, and nearly square shaped, articulating with each other in the middle line, as in *Lepidosteus* and *Polypterus*, there being no supraoccipital interposed as in most *Telostei*. In advance of them, and forming a large part of the vault of the cranium, we have the frontals (11). From the frontals we pass on to the single nasal bone (15), which forms the prominent point of the snout above the mouth. Opposite this prominence the nasal bone is notched on each side for the olfactory opening, and from the prominence itself pass beautifully radiating lines over the whole surface of the bone. On each side of the nasal bone is situated the pre-frontal (14, figs. 3 and 4,) which completes the olfactory notch of the former bone into a nearly round opening. It articulates internally with the nasal, below with the intermaxillary, and behind with the bone to be next described as post-frontal. Externally, it forms the anterior part of the orbital margin of the cranial shield.

The bones (12) behind the pre-frontals, and forming the posterior part of the orbital margin of the cranial shield, I can imagine to be nothing else than the post-frontals, though they are certainly situated rather far forwards. They

articulate in front with the pre-frontals, behind with the mastoids, internally with the principal frontals, which they thus completely exclude from the orbital margin.

The *mastoid* (8) is very distinct, forming an elongated plate external to the parietals, and articulating with them, and with the frontal and post-frontal bones. Externally it is in contact with the facial bones; and close beneath its anterior part may be seen the attachment of the suspensory apparatus of the lower jaw. Behind it forms, along with the parietal, the posterior margin of the cranial buckler, and is succeeded in that direction by the suprascapular (50. fig. 2).

Face.—Recurring to the profile view of the head, fig. 2, we observe that the *superior maxillary* bone (21) is of great strength. Posteriorly it is very broad, and overlaps the lower jaw near its articulation. Anteriorly it becomes, in the orbital region, suddenly narrowed to a somewhat slender point, because of its upper border being suddenly beveled off to form a curved margin, along which the suborbital bone (73) articulates. The narrow extremity proceeds forwards below the orbit to come in contact with the intermaxillary, and the external surface is seen in fig. 1 to be ornamented with delicate striæ, which run parallel with the upper and hinder border of the bone. The posterior-inferior angle, overlapping the articulation of the lower jaw, is covered, however, with a minute tuberculation.

The *intermaxillary* (22) is a little bone bearing teeth, placed at the front of the orbit, and beneath the snout and nasal openings. It is in connection above with the pre-frontal; internally and above with the nasal, behind with the superior maxillary; whether it articulates also with its fellow of the opposite side is not seen distinctly. Externally it continues the orbital margin of the cranial shield downwards and somewhat backwards, to which shield it is immovably articulated.

The *lower jaw* (L, J) is stout and curved, the curve having the convexity downwards. Of how many pieces it is composed it is impossible to determine, though there are evidently at least two, viz.—dentary and articular. Its outer surface is covered with a beautiful minute tuberculation, seen in fig. 1.

Teeth may be observed on the superior maxillary and intermaxillary bones, and also on the lower jaw. They are of two descriptions, large ones of about two lines in length, alternating with smaller ones, of about half that size. These teeth are sharply pointed, and of a slender conical form; their external surface is shining and smooth, their transverse section round, and the pulp cavity large. Fig. 7 represents a portion of the edge of a lower jaw, with one large and several smaller teeth attached.

Opercular apparatus.—The *operculum* is an oblong, and somewhat rhomboidal plate, its anterior, superior, and posterior inferior angles being rather acute. Its upper margin is situated close beneath the suprascapular and mastoid bones, and its long axis is directed at a very considerable angle downwards and *back-*

wards, by reason of the wide extent of the gape, and the consequent very posterior position of the articular end of the lower jaw.

The *suboperculum* (36) is nearly square, but its posterior margin is longer than the anterior one, which though also directed somewhat backwards, is not so much so as the anterior margin of the previous bone, with which it consequently forms a slight angle.

I have looked in vain for any very distinct representatives of the *preoperculum* and *interoperculum* of other fishes. On referring to fig. 2, it will be seen that, in front of the operculum, follows the triangular plate (*g*) covering the cheek, a small portion of the suspensory apparatus of the lower jaw being exposed between them above; while in front of the suboperculum follows the hinder border of the upper jaw. If present, the bones in question must then be reduced to very narrow laminæ, so as to render their recognition very difficult, at least in the state of preservation in which these fossils usually occur.

Hyoid bone and branchiostegal rays.—What I consider to be the posterior part of the body of the *hyoid*, may be seen in fig, 1, *h*, with four of the branchiostegal plates behind it. The latter are also shown in fig. 2, the lower ones being broken off behind, so as to expose the underlying coracoid (52). None of my specimens show the branchiostegal rays in complete series, so that their number cannot be exactly ascertained. In one example I counted at least twelve; and there can be no doubt that there were a great many more, probably one-half as many more again. They take the form of oblong imbricating plates, whose posterior margin is broader than the anterior one. One of the upper plates in fig. 2. measures 9 lines in length, and 5 lines in breadth, at the hinder border.

In front of the operculum, and above the posterior part of the superior maxillary bone, is a triangular plate (*g*) with the acutely-pointed apex directed backwards in the angle between the two bones just named. Between it and the operculum above may be seen, as already remarked, a portion of the suspensorium of the lower jaw exposed. This plate may be a member of the "Gesichtspanzer," representing the cuirass of small plates behind the eye and in front of the operculum of *Lepidosteus*. It will also be seen to have exactly the same relative position to the other bones of the head, as the bone covering the cheek and masticatory muscles in *Polypterus*, and which by AGASSIZ and MÜLLER is reckoned as "preoperculum;" by HUXLEY as "supratemporal."

In front of this plate is a sickle-shaped suborbital bone (73), surrounding the posterior and lower margin of the orbit, and likewise fitting on to the lunated anterior-superior part of the superior maxillary bone. There are doubtful indications of another, just behind the upper margin; the rest of the chain are crushed inwards, and concealed beneath the ironstone matrix. On examining some other heads, however, it seems probable that the eye was completely surrounded, as in *Lepidosteus*, by a chain of such narrow ossicles.

Shoulder-girdle.—Above the operculum, and behind the mastoid and parietal

bones, may be seen a well-marked *suprascapular* (50). Between it and the hinder edge of the cranial shield may be seen faint traces of what I should consider a supratemporal ossicle. The suprascapular is followed by a flat and elongated *scapula* (51), which reaches down, at least, to the middle of the hinder edge of the suboperculum. It is in its turn followed by a large *coracoid* (52), its lower extremity being seen in fig. 2, exposed by the breaking off of some of the branchiostegal plates. This bone, the upper and larger part of which is flattened laterally with a convex posterior margin, forms below, near the origin of the pectoral fin, a prominent angle, being suddenly bent on itself inwards towards the middle line of the throat. Succeeding the lower part of the coracoid in front is a triangular plate (*f*), with the acutely-pointed apex directed forwards, and folded on itself laterally, so as to present both a lateral and an inferior or jugular surface. This is certainly the equivalent of the plate succeeding the lower end of the coracoid in *Polypterus*, and which is also seen in many of the *Crossopterygidæ* of the Old Red Sandstone. It is, in the specimen from which fig. 2 was taken, partly exposed by the breaking away of some of the branchiostegal plates.

These are, then, all the facts that I have been able to elucidate with any certainty regarding the structure of *Pygopterus Greenockii*. Before, however, proceeding to any comparison with the recent Ganoids, let us examine a little into the conformation, as far as can be made out, of the head in certain other upper Palæozoic genera, viz., *Amblypterus* and *Palæoniscus*.

I have not been able to obtain any further description of the bones of the head in *Amblypterus* and *Palæoniscus* beyond that of AGASSIZ in the second volume of the "Poissons Fossiles." The points to which attention is there directed are:— The projection of the snout in front of the mouth by reason of an expansion of the frontal and nasal bones (ethmoïde); the great extent of the gape; the strength of the jaws, which are furnished with teeth "*en brosse;*" that the orbit is bounded below by a series of suborbitals; that the branchiostegal rays form a series of flattened plates between the two halves of the lower jaw. The opercular apparatus is described as being formed of the usual four pieces, of which the operculum is largest. Of the scapular arch three bones are mentioned, viz., the suprascapular, scapular, and coracoid (*humerus*). The nasal projection is described as wanting in *Amblypterus*.

Not having been myself able to notice any anatomical difference between heads of *Amblypterus* and *Palæoniscus* in such species as have come under my notice, and as my knowledge is principally derived from specimens of *Amblypterus* from the shales of Wardie, I will assume that genus as the type, believing, however, that all statements regarding its general cranial structure hold equally good with regard to *Palæoniscus*. After having examined a great number of specimens of *Amblypterus** from the above-named locality, I have, in figs. 10 and 11, given a

* Principally *A. punctatus* (AGASSIZ).

sketch of the various bones which I have been able to distinguish. On looking at fig. 10, the first thing which strikes the eye is the nearly exact resemblance presented by the outlines of the bones to those in *Pygopterus*, if we take the above described *P. Greenockii* as an example of the genus. Of the bones of the shoulder-girdle, the suprascapular (50) is readily distinguished as a triangular, somewhat convex, plate at the hinder part of the skull above the operculum. To this succeeds the elongated scapula (51), and then the coracoid (52), which, as in *Pygopterus*, forms an angle at the pectoral fin where it bends inwards on the under surface of the throat towards its fellow of the opposite side. We have also the triangular pre-coracoid plate, which, meeting in the middle line with its fellow of the opposite side, forms with it a pointed process, passing forwards the two lateral series of branchiostegal plates (44).

The opercular apparatus is conformed as in *Pygopterus Greenockii*, the operculum (35) being narrow and of an oblong rhomboidal shape, the suboperculum (36) nearly square; while the same difficulty is experienced in finding a distinct pre- and sub-operculum, so that those bones, if present, must have been very narrow. The *superior maxillary* bone (21), though not so strong as in *Pygopterus*, has essentially the same form, consisting of a stout oblong plate, its posterior margin sloping downwards and backwards, and its superior margin beveled off in a semilunar manner below and behind the orbit. The *lower jaw* (L, J) is stout, and beautifully tapering from its posterior articular extremity towards the symphysis; the gape extends very far back; the teeth in both jaws are minute, and only to be studied by means of a lens; then they are seen to be smooth, and of an acutely-pointed slender-conical form. The *branchiostegal plates* (44, fig. 11) are fourteen on each side, with an azygos one in the middle, immediately behind the symphysis of the lower jaw. The azygos plate, and the anterior one of the lateral series on each side, are of a rhomboidal form; the rest are more narrow, oblong plates, with the hinder margin rather broader than the front. They seem also to be slightly shorter in the middle than at each end of the series. At the lower and posterior border of the orbit may always be seen a sickle-shaped *suborbital* (73), and the examination of some specimens, a little better preserved than others, leads me to the conclusion that the eye was completely encircled by a chain of similar ossicles, but which, however, from the crushing to which the heads have been subjected, cannot be counted or definitely sketched. Behind the orbit, and above the upper jaw-bone, are also one or more plates covering the cheek.

Owing to the crushed state in which the heads of the smaller fishes occur, I have no specimen in my collection which shows the bones of the cranial shield so beautifully as they are displayed in the head of *Pygopterus*, figured Plate XLV., figs. 2, 3, 4. The examination of several specimens in a better or worse state of preservation shows, however, that their cranial bones were essentially similar to those of the last-named fish, the whole cranial shield being apparently a little

broader and shorter proportionately. The square-shaped *parietals*, the more elongated *frontals*, the *mastoids*, the *post-frontals* forming the posterior, and the *pre-frontals* the anterior part of the orbital margins of the shield, may all be distinctly recognised. The same line passes along the parietal and frontal on each side which I have noticed in *Pygopterus*. The nasal bone certainly projects forward over the mouth, but is invariably so crushed as to render it almost impossible to tell with any certainty if it be a single bone, as in *Pygopterus*, or double, as in *Lepidosteus* and *Polypterus*. For my part, I should incline to the former opinion.

The head may now be seen to be very similarly constructed in *Pygopterus* and in *Amblypterus*. On comparing the head of *Pygopterus* with that in the recent *Lepidosteus* and *Polypterus*, it will be seen that the gape is still wider, and the articulation of the lower jaw still further back than in the latter genus, so that the suspensory and opercular apparatus are directed obliquely downwards and backwards.

In its relation to the superior maxillary bone, the orbit is still more anteriorly situated than in *Polypterus*, being placed over the anterior part of the upper jaw, whereas in *Lepidosteus* the projection of the snout and the length of the suspensorium carry both jaws forward altogether in front of the eye. The superior maxillary bone is simple, as in *Polypterus*—not divided into a number of pieces, as in *Lepidosteus*. As in both *Lepidosteus* and *Polypterus*, the intermaxillary bone is immovably articulated to the front of the cranium, only a limited amount of motion being allowed to the superior maxillary. There are a greater number of separate bones exposed on the top of the cranium than in *Polypterus*; but the nasal bone is single, thus differing from the nasal both in *Polypterus* and *Lepidosteus*, which is double. The post-frontal is seen on the surface of the skull, but again it is doubtful whether any representatives of the supra or paroccipitals take part, as in *Lepidosteus*, in the formation of the cranial shield. The branchiostegal membrane was strengthened by rays instead of by two "jugular" plates, as in *Polypterus*, but again these rays differ from those of *Lepidosteus* in being many in number, and enamelled on their surfaces.

The bones of the shoulder-girdle are the same as those in *Polypterus*. Lastly, the head of *Pygopterus* differs from that of *Polypterus* in the absence of the spiracle, and the row of ossicles associated therewith. The only undoubted members of the system of superficial facial bones are the suborbitals and the plate covering the cheek; the presence of supratemporals is doubtful; and I have seen no trace of the "*os mobile du nez*," though this, judging from its small size in *Polypterus*, may readily become indistinguishable in a fossil specimen.

Turning now to *Eurynotus* (AG.), a genus not uncommon in the Scottish Carboniferous strata, but rare elsewhere, we find that it has many points of resemblance to the two fossil genera already noticed, but also very many of decided difference.

In his "Poissons Fossiles," AGASSIZ has said very little about the bones of the head of *Eurynotus*, merely noticing the narrow shape of the operculum and sub-operculum, and the proportionally great height of the latter, also observing a triangular plate which he considered as a suborbital bone; lastly, that the teeth were small and obtuse. I regret that my knowledge of the osteology of the head of this fish is very imperfect; what I do know of it is derived from specimens of *Eurynotus fimbriatus* from Wardie, and in figs. 8 and 9 I have sketched the forms of the various bones I have been able to make out.

Regarding the *cranium proper* I can say nothing, the specimens being too much crushed.

The *operculum* is very small in proportion to the size of the fish, and is of a short oblong form, with the posterior-superior and anterior-inferior angles somewhat rounded off. The *suboperculum* is larger, but also oblong in form; its upper margin is concave; the posterior-inferior angle is very much rounded off; a diagonal line from the posterior-superior to the anterior-inferior angle divides the external surface into two areas; and of these the upper one is ornamented by ridges radiating from the anterior-inferior angle of the bone, while the lower area is marked by concentric ridges running parallel with the rounded posterior-inferior margin. In front of the two plates just described is another elongated one which certainly seems to represent a preoperculum. The superior maxillary bone (22, figs. 8 and 12) is most decidedly different from that in *Amblypterus* and *Pygopterus*, consisting of an elongated triangular plate with the apex directed forwards. The lower margin, which is the longest, is seen to be garnished along the whole of its inner edge by small rounded polished teeth, which resemble nothing so much as small and somewhat flattened grains of shot. They are placed irregularly together, the largest ones being about the middle of the series; about one-third from the anterior extremity of the bone they suddenly become excessively minute, and at the same time are not placed so close together. Some are observed to be rather narrowed at the base, but this is not universal. The border next in length is the upper, which slopes towards the apex in front. The short side is posterior, and fits on in front of the plate which I have considered the preoperculum. The external surface of the superior maxillary bone is sculptured by coarse ridges which run parallel to the posterior and upper margins.

Lying on the same piece of stone with the superior maxillary bone, from which fig. 12 has been taken, is a flat irregularly shaped bone (fig. 13), which must appertain to the *palate*. It is undefined at one extremity, and passes into two irregular processes at the other. A large portion of its surface is completely covered by the rounded teeth already described, and which are indeed so thickly placed together, that in many instances their bases assume a polygonal form. The part of the bone bearing the teeth is marked by three longitudinal ridges bounding two shallow furrows. Two of those ridges form the margins of the

tooth-bearing part, and of those two one also coincides with the edge of the entire bone; the third passes along midway between them, and carries the largest teeth.

The *lower jaw* (fig. 8, L, J) is moderately stout, and between its two halves are seen branchiostegal plates exactly resembling those of *Amblypterus*. There is, namely, a median lozenge-shaped plate behind the symphysis of the jaw, and the first lateral one on each side is nearly regularly rhomboidal, being very much broader than those which succeed it posteriorly. The number of these plates I have not ascertained; in one example I counted at least 10, but there must be many more.

Of the *pectoral arch*, the only bones to be seen with certainty are the *coracoid* and the plate succeeding it below and in front, which latter is proportionally larger than in *Amblypterus*. The coracoid resembles that of *Amblypterus* in general shape; but its upper part is apparently rather slender, whereas the lower reflected part is very broad. Traces of a *scapula* are seen in one specimen, and also between and behind the scapula and coracoid a semilunar ossicle resembling one of the plates found in a similar situation in *Polypterus*.

On comparing these few facts regarding the head of *Eurynotus* with those more completely elucidated in *Pygopterus* and *Amblypterus*, we see that the first-named genus decidedly differs in the form of the opercular apparatus, and of the superior maxillary bone, in the shape and arrangement of the teeth, and in the smaller extent of the gape, so that the suspensory and opercular bones do not require to be directed so very much backwards towards the articulation of the lower jaw.

On the other hand, *Eurynotus* agrees with *Amblypterus* in the form and arrangement of the branchiostegal rays, in the form of the scales, and in the structure of the fins* and tail, the latter being typically heterocercal, defended along its upper border by a row of V-shaped scales, while the sides of the vertebral prolongation are covered by elongated ones of a lozenge shape. The fins were all furnished with large fulcral scales along their anterior margins, which, in the dorsal fin of a specimen in the museum at St Andrews, may be seen to be arranged in a *double series*. This double arrangement of the fin-fulcra is also recorded of *Palæoniscus* and *Acrolepis* by MÜLLER (" Ganoiden," p. 152).

Agreeing with MÜLLER that the division between the "*Lepidoids*" and "*Sauroids*," the two families in which *Amblypterus* and *Pygopterus* have been placed respectively by AGASSIZ, is artificial, we must class those two genera and their immediate allies (*Palæoniscus*, *Catopterus*, *Acrolepis*, &c.) in one family of *Palæoniscidæ*, as has been done by VOGT ("Zoologische Briefe," s. 133). Then, accepting meanwhile the great divisions of the Ganoid order proposed by HUXLEY ("Memoirs of Geol. Survey, Decade X.," 1863), these fishes must come in under the sub-order *Lepidosteidæ*, characterised by the possession of non-lobate paired fins, rhomboidal scales, and branchiostegal rays; and under the family *Lepidotini*,

* Except in the large size and peculiar form of the dorsal fin in *Eurynotus*.

in which the superior maxilla is formed of one piece, and the branchiostegal rays are many and enamelled. Here they must form a distinct sub-family of *Palæoniscidæ*, equivalent to the "*Lepidoidei heterocerci*" catalogued by Sir PHILIP EGERTON in the "Quart. Jour. of Geol. Society," (vol. vi. 1850), but with the addition of *Pygopterus*, *Acrolepis*, and some other allied genera, generally classed as "*Sauroids*," the family being characterised by their wide gape, many rayed fulcrated fins, and by their completely heterocercal tail, the upper border of which is set with a row of imbricating V-shaped scales.

But the last question is, whether *Eurynotus* is also to be considered a member of the *Palæoniscidæ*, or to be transferred to the *Pycnodonts*, which it resembles in the rounded crushing palatal and other teeth. This question has already been discussed by Sir PHILIP EGERTON in the paper above referred to; and the conclusion to which he there arrives, taking the form and structure of the scales and fins into account, is, that *Eurynotus* ought to retain its place among the "*Lepidoidei heterocerci*." But in a paper very recently read before the Geological Society of London, Dr J. YOUNG has removed it from its old associates, and placed it, together with *Platysomus*, the *Pycnodonts*, and two new genera, *Amphicentrum* (YOUNG), and *Mesolepis* (YOUNG), in his family of *Lepidopleuridæ*. I regret that my knowledge of the last-named fishes is not sufficient to enable me to venture an independent opinion as to the relations to them of *Eurynotus*. As regards the configuration of the scales, the structure and general form of the fins, except in the case of the dorsal, *Eurynotus* certainly bears a very strong resemblance to *Amblypterus* and *Palæoniscus*. But the dentition and the shape of some of the bones of the head distinguish it so much from the two last-named genera, that it is not impossible, that, when we come to know more about the cranial structure of *Eurynotus* and that of the apparently allied genus *Mesolepis* (YOUNG), Dr YOUNG's ideas will be found to be substantially correct.

Explanation of Plate XLV.

The various bones are designated and numbered according to the nomenclature used by Professor OWEN in his "Lectures on Comparative Anatomy."

Fig. 1. *Pygopterus Greenockii* (Agassiz), one-half natural size.
D, Dorsal fin; C, Caudal; A, Anal; V, Ventral.

Fig. 2. Head of another specimen seen from the side, diminished one-third.
L, J, Lower jaw; N, Nasal opening; *f*, Pre-coracoid plate; *g*, Triangular cheek-plate.
15, Nasal bone; 21, Superior maxillary; 22, Pre-maxillary; 35, Operculum; 36, Suboperculum; 44, Branchiostegal rays or plates; 50, Suprascapular; 51, Scapular; 52, Coracoid; 73, Suborbital.

Fig. 3. Upper surface of the cranial buckler of the same specimen represented in fig. 2 natural size.

N, N, Nasal openings.

7, 7, Parietal bones; 8, 8, Mastoids; 11, 11, Frontals; 12, 12, Post-frontals; 14, 14, Pre-frontals; 15, Nasal.

Fig. 4. Front view of snout of same specimen.

N, N, Nasal openings.

14, 14, Pre-frontals; 15, Nasal; 22, 22, Pre-maxillaries.

Fig. 5. Scales of *Pygopterus Greenockii*.

A, Scale from the anterior part of the flank of the fish magnified two diameters, showing concentric lines of growth, the external layer of ganoine having scaled off.

B, Outline of a scale from the hinder part of the flank.

C, Scale from anterior part of the flank, magnified three diameters, and showing part of the external striated layer of ganoine.

Fig. 6. Exposed surfaces of scales from posterior part of flank, magnified two diameters. On one of those the external striation is represented.

Fig. 7. Portion of edge of lower jaw, showing two sizes of teeth.

Fig. 8. Diagram of side of head of *Eurynotus*. The numbers apply to the same bones as in fig. 2.

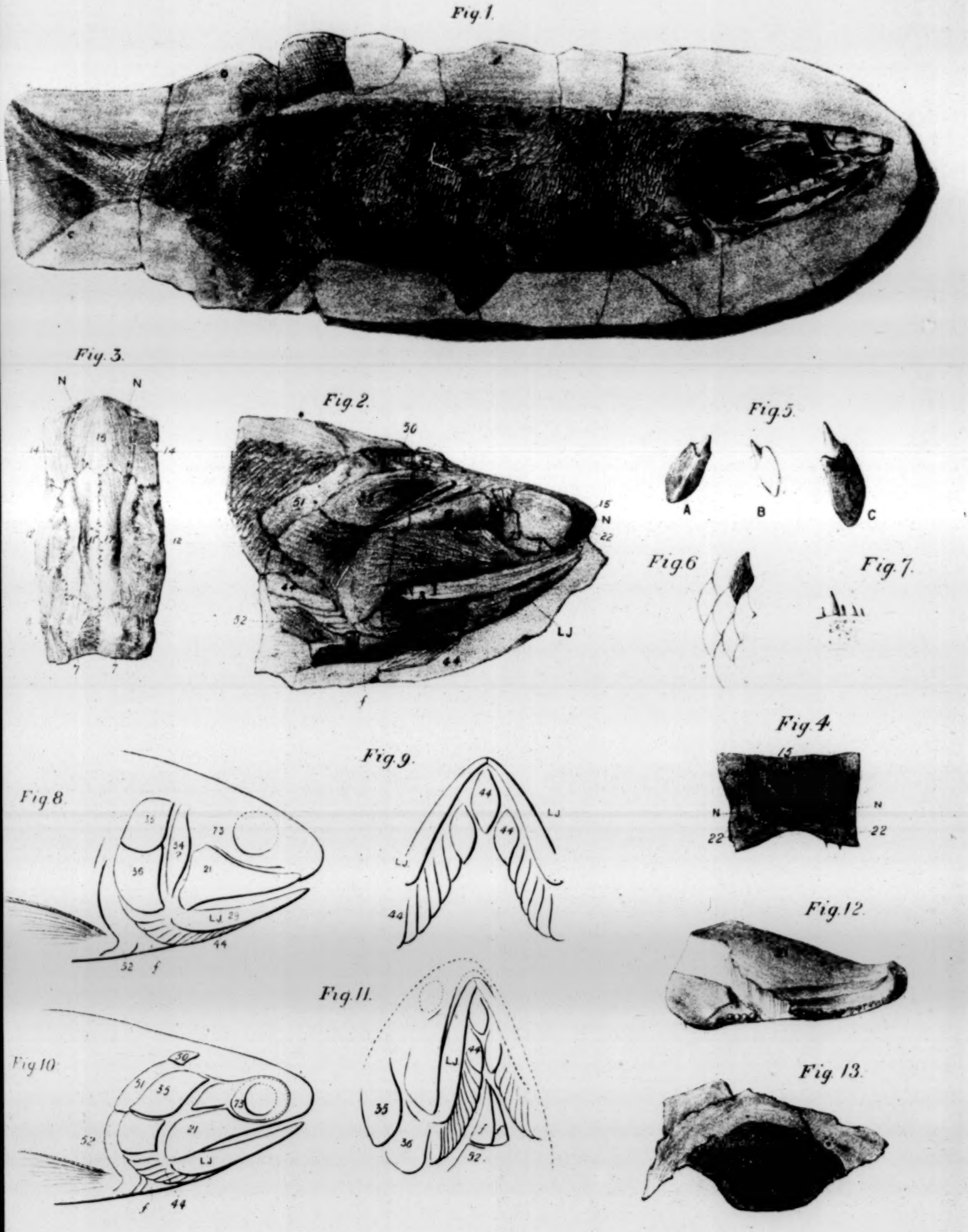
Fig. 9. Diagram of arrangement of branchiostegal rays in *Eurynotus*.

Fig. 10. Diagram of side of head in *Amblypterus*.

Fig. 11. Diagram of head of *Amblypterus*, seen obliquely from the side, and from below.

Fig. 12. Superior maxillary bone of *Eurynotus*, seen from the inside; natural size. From South Queensferry.

Fig. 13. Palate (?) bone of *Eurynotus*, with crushing teeth. From the same locality.



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XLVIII.—*On the Physiological Action of the Calabar Bean* (Physostigma venenosum, *Balf.*). By Thomas R. Fraser, M.D., Assistant to the Professor of Materia Medica in the University of Edinburgh. Communicated by Professor Christison, M.D., D.C.L., V.P.R.S.E.

(Read 17th December 1866.)

In 1855, the Professor of Materia Medica in the University of Edinburgh, in a paper read before this Society, directed the attention of physiologists to some of the remarkable properties of the Calabar bean.* In 1862, I presented a graduation thesis to the University of Edinburgh on the "Characters, Actions and Therapeutic Uses of the Ordeal Bean of Calabar." The principal results I had obtained at that time were that this substance causes death by either syncope or asphyxia, the latter being due to an effect on the spinal cord and on the respiratory centres; that the symptoms resemble those of cardiac or pulmonary embarrassment, according to the quantity of the poison administered, and to its rate of absorption; and, also, that the topical application of this agent to the eyeball, or to its neighbourhood, produces a marked and rapid contraction of the pupil and various disturbances of vision.† Since then, and more especially because of the peculiarity of the last of these conclusions, a lively interest has been taken in this substance. Its actions on the eye have been investigated by nearly all the leading ophthalmologists of Europe and of America, and its general physiology has occupied the attention of many distinguished students of biology. Nor have these labours been barren of practical results. Ophthalmic medicine has adopted this agent as one of its important remedies, and there can be little doubt that general medical practice will soon include in its Pharmacopœia a drug of so great energy.‡

The present investigation was undertaken for the purpose of extending and supporting my previous results, with some of which subsequent observers have disagreed; but I purpose to take an opportunity of examining these discrepancies with some detail in a different place. The effects which follow the topical application to the eyeball will be merely alluded to in this paper, as this portion of the subject has not been completed. Enough has, however, been done to convince

* Proceedings of the Royal Society of Edinburgh, vol. iii. p. 280; and Monthly Medical Journal, vol. xx., 1855.

† Edinburgh Medical Journal, 1863, and pamphlet.

‡ Since this sentence was written the *Physostigmatis Faba* has been admitted into the edition of the "British Pharmacopœia," published in 1867.

me of the insufficiency of the views hitherto advanced, and to suggest the advisability of extending my observations.

Preparations.

In 1863, I separated from the kernel, from the spermoderm of the bean, and, also, soon after, from the excrement of a lepidopterous insect which feeds on the kernel,* an amorphous active principle, possessing the general properties of a vegetable alkaloid, for which I proposed the name Eserinia, derived from Eserĕ, the usual name of this ordeal-poison at Calabar; and with it a few experiments were made, some of which have been published. Shortly afterwards, I succeeded in obtaining this alkaloid in, apparently, a state of greater purity, and as a crystalline substance, to which I gave the name Eseria. A crystalline acid, having a similarity to, and being probably identical with, tartaric acid, was also obtained from the kernel at that time. In the present investigation, however, an extract, prepared by acting on the finely pulverised kernel with boiling alcohol (85 per cent.), has been used. This preparation contains a considerable proportion of fatty matter, which prevents its complete solution in water; and, as the division into separate doses of a mere watery suspension would lead to many inaccuracies, it was found necessary to weigh the requisite quantity, separately, for the majority of the experiments. This extract is hygroscopic, which further required that it should be dried and kept in an exsiccator in order to ensure an unvarying preparation.†

Subjects of Experiment and Comparative Effects of Doses.

With few exceptions, the experiments were made with the common frog (*Rana temporaria*), birds, and various mammals. It was found that fatal results were produced with the smallest quantity on birds; and that the largest doses, in proportion to weight, were required by amphibia. A dose of one-sixteenth of a grain proved rapidly fatal to a pigeon weighing nine ounces and three-quarters; whereas a frog, which weighed 726 grains, has recovered from three grains of extract—a quantity sufficient to produce death in a dog of average size.

A. ACTION THROUGH THE BLOOD.

As I have already, in a previous paper, described with considerable detail, the general symptoms which follow the administration of physostigma, it will be unnecessary to give them here. It has also been shown, on the same occasion, that the more rapid the absorption of the poison the more quickly are fatal effects produced, and that the active principle may be absorbed by any living

* On the Moth of the Esere, or Ordeal Bean of Old Calabar. The Annals and Magazine of Natural History, May 1864, pp. 389–393.

† The varying potency of an extract possessing the property of absorbing moisture may unfit it for therapeutic purposes, but the tincture I have already recommended (*op. cit.* sect. iii.) will prove a sufficient substitute, and it has the great advantage of constancy of strength.

tissue. From the following experiment it is proved that prolonged digestion with gastric juice does not impair the energy of Calabar bean:—

Experiment I.

A gastric fistula was formed in a healthy dog, and, some days afterwards, and while the animal was in good health, 500 grains of gastric juice were withdrawn from the stomach. Four hundred grains of this were mixed with half a grain of extract of physostigma, received in a flask with an arrangement to impede evaporation, and placed in a water-oven at a temperature of 98° F. The digestion was continued for twenty-four hours, when the fluid was placed in a capsule and evaporated at 85° F. The resulting extract was finely pulverised, heated with alcohol of 85 per cent., filtered, and again evaporated to dryness. Contact with distilled water removed an acid fluid, which was made alkaline by excess of magnesia, and agitated in a bottle with chloroform. The chloroformic solution was removed by a separating funnel and evaporated, and the resulting brown extract was suspended in distilled water. A drop of this was applied to the conjunctiva over the right eyeball of a rabbit, whose pupil, before the experiment, measured $\frac{12}{60}$ths × $\frac{16}{60}$ths of an inch. In eight minutes, the pupil was $\frac{8}{60}$ths × $\frac{12}{60}$ths; in fifteen minutes, $\frac{6}{60}$ths × $\frac{8}{60}$ths; in twenty minutes, $\frac{4}{60}$ths × $\frac{5}{60}$ths, and it continued in this contracted condition for many hours. The remainder of the fluid was injected under the skin of a young pigeon, and caused its death in eight minutes.

Several small pieces of hard-boiled white of egg were placed in a flask with the remaining 100 grains of gastric juice, and digested under exactly the same conditions, and at the same time, as the extract of physostigma. They were found to be completely dissolved in less than ten hours. There could, therefore, be no doubt as to the activity of the gastric juice which had been employed.

This merely confirms the result before obtained, of fatal effects following the introduction of the poison by the digestive system.

I believe that BRINTON first demonstrated that a poison which had been administered by the blood may be excreted by the stomach and intestines.* This was proved with tartar emetic; and, more recently, TAYLOR has published evidence showing that arsenic also may appear in the stomach although it had not been administered by the alimentary canal.† I took an opportunity of examining if a similar event occurs in poisoning with physostigma.

Experiment II.

Five grains of extract, suspended in water, were injected into the right jugular vein of a dog, and caused the death of the animal in eleven minutes. The stomach was immediately removed, and its contents, along with some of its mucous coat, obtained by scraping, were partially dried at a low temperature, and then boiled with successive portions of spirit (85 per cent.) acidulated with tartaric acid. The tincture was concentrated by distillation, and then evaporated to dryness. The extract was treated with distilled water, filtered, and agitated with ether until the fatty matters were removed. The remaining watery solution was made alkaline by the addition of carbonate of sodium and shaken with ether; and the ethereal solution was distilled. A yellowish, alkaline, amorphous residue was obtained, weighing three-fifths of a grain, and having a disagreeable animal odour. A minute portion of this extract was mixed with two

* Cyclopædia of Anatomy, article "Stomach;" Lancet, 1853, vol. ii. p. 599; and Lectures on the Diseases of the Stomach, 2d edit., 1864, p. 54.

† Guy's Hospital Reports, vol. vi. p. 397.

drops of distilled water and applied to the conjunctiva of a white rabbit, in the presence of my friend, Dr CRUM BROWN. Before the application, both pupils had a diameter of $\frac{13}{60}$ths of an inch, in a full light. At first a little irritation was caused. In thirteen minutes, the pupil had contracted to $\frac{5}{60}$ths, and in eighteen minutes to $\frac{3}{60}$ths; the other pupil still remaining at its original diameter of $\frac{13}{60}$ths. This extreme contraction continued for upwards of an hour; but in two hours the pupil was $\frac{7}{60}$ths, and by the following morning it had resumed its original diameter.

It would, therefore, appear that physostigma, when administered by a vein, finds its way into the stomach—a method of poison-excretion which has been established in the cases of antimony and arsenic.

Although this investigation has for its principal aim the determination of the exact method in which physostigma acts, and the demonstration, as far as possible, of the histological structures which it influences, it may be necessary to describe, at this place, the general symptoms which follow the administration of a poisonous dose. In the case of mammals, I have already entered fully into this subject in a previous paper, from which I extract the following descriptions:*—

"When a *small* fatal dose is administered to one of the lower animals, a train of symptoms is produced usually in the following order:—A slight tremor is first seen, especially at the posterior regions, and this extends forwards to the anterior extremities and the head. The limbs yield immediately afterwards, the posterior becoming generally first paralysed, and the animal lies extended in a state of almost complete muscular flaccidity. A few attempts may be made to recover the normal position, but they are usually ineffectual. The bowels, in most cases, are evacuated, and urine is passed. The pupils generally *contract*; as the symptoms advance, the respiration becomes slow and irregular, with a distinct stertor accompanying both inspiration and expiration, and frothy mucus escapes from the mouth. Muscular twitches occur, and often continue after respiration has ceased. Reflex action cannot be produced by either pinching or pricking the skin. By-and-by the eyelids do not contract when touched or even when the eyeball is pricked. On lifting by the ears, the limbs hang inertly, and the only sign of life is an occasional gasping inspiration, which also soon ceases, and the animal appears dead.

"Consciousness is preserved during the whole time, until the power of expression is lost. During incomplete paralysis, proofs of sensation may be obtained by pinching the ears or pricking the skin. Immediately after death the pupils dilate.

"On opening the body the various muscles which are cut contract. The diaphragm and muscles of the extremities may be excited to action by pinching the phrenic and sciatic nerves, and the contractility of the muscles generally is retained for some time after death. The heart is found acting regularly, and the intestines exhibit distinct vermicular action. The heart may continue its action

* *Op. cit.* sect. ii.

for one hour and a-half after death. Its chambers usually cease to contract in a definite order, the left auricle first losing its spontaneous action, then the right and left ventricles, and, after an interval, the right auricle. The large veins in the thorax are found distended. . . . The lungs are engorged—in two experiments this had proceeded to such an extent that detached portions sank in water. . . .

"When a *large* fatal dose of the kernel is administered, the hind limbs almost immediately yield, and the animal falls. It lies flaccid, and in any position, on the table, and exhibits muscular power only by a few twitches. The pupils contract; in a few cases fluid escapes from the nostrils and mouth, and the lachrymal secretion is increased. Reflex action cannot be produced by irritation, and the respirations, after a few gasps, cease.

"The pupils dilate immediately after death. On opening the body, muscular twitches occur. . . . The heart is found distended and passive; irritation, however, produces contraction for about ten minutes after death. The vermicular action of the intestines is very much diminished, and can scarcely be observed. . . . The mesenteric arteries and veins may be readily distinguished by the colours of their contents."

The following will serve to illustrate the symptoms with frogs:—

Experiment III.

Three grains of extract of physostigma, suspended in twenty minims of distilled water, were injected, by WOOD's syringe, into the subcutaneous cellular tissue at the back of a light-coloured frog, weighing 430 grains. For four minutes it appeared perfectly unaffected and jumped about normally; after which time some increase occurred in the respiratory irregularity which is always found in frogs. In seven minutes, the respiratory movements of the chest had ceased; but those of the throat continued for other four minutes (eleven after injection). About this time, the movements of the animal were sluggish; the fore legs gradually began to separate until they no longer supported the chest and head; and the posterior extremities were affected in a like manner, and soon after lay extended and flaccid. Weak voluntary movements, however, continued until fifteen minutes after the exhibition of the poison; and, for some time after this, irritation demonstrated the continuance of reflex power. In half an hour, the skin of the frog had undergone a marked change, having become of a dark brown colour. Although now apparently dead, it was not, in a strict physiological sense, really so. Motor nerve-conductivity was retained for many minutes longer; the diastaltic function was not abolished, and, hence, it was possible to show that afferent nerve-conductivity also continued; and the muscular tissue, for many hours, contracted when stimulated, and, in the case of some of the heart chambers, spontaneously and successively did so, for a shorter period.

Into all these, and many other points, it is necessary to enter with detail, and several of them may be overtaken in a somewhat connected manner by examining the cause of what is the most prominent, as well as one of the earliest, of the phenomena described. This is obviously the condition of gradually increasing paralysis.

Action on the Voluntary Muscles.

The peculiar successive tremors, which are observed in warm-blooded animals, at first sight suggest that the paralysis caused by Calabar bean is due to an affection of the muscular system; and the condition of general flaccidity, which so rapidly follows its administration to frogs, appears to favour, as it certainly does not contradict, this opinion. Without pretending that such was the order followed in this investigation, it will, as a matter of convenience, be advisable to examine, in the first place, the effects which are produced on voluntary muscles.

Experiment IV.

A full-grown active rabbit had injected into the subcutaneous tissue of its right flank, three grains of extract, suspended in eleven minims of distilled water. Tremors occurred in two minutes; the anterior extremities soon after yielded; and, in four minutes and thirty seconds, the animal fell, the muscular trembling having increased in vigour and having become general over the body. Respiration ceased in five minutes after the injection, but muscular tremors continued during other three minutes. When the thorax was opened the heart was found dilated and passive. Twenty-four minutes after the administration, galvanic stimulation of the sciatic nerves caused powerful muscular contractions; within thirty-six minutes, these nerves were completely paralysed, though application of the electrodes to any of the voluntary muscles produced marked contractions. These contractions became gradually weaker, but could be distinctly excited until one hour and thirteen minutes after the poison had been exhibited.

The general result in all the other experiments which were performed on warm-blooded animals was the same. Muscular contractility remained after destruction of the function of motor nerves; and this also occurred, in even a more marked manner, with frogs.

Experiment V.—(*Temperature of Laboratory about* 53° *F.*)

By means of Wood's syringe, I injected three grains of extract, in fifteen minims of distilled water, into the lower portion of the abdominal cavity of a frog which weighed 473 grains. The usual phenomena quickly occurred. In sixteen minutes, the sciatic nerve and the neighbouring muscles of the left thigh were exposed and found active.

The muscles were now of a very blue colour, quite distinguishable from their normal appearance; and this colour change was discovered in the serous and fibro-serous tissues also. In about four hours, motor nerve-conductivity was universally destroyed. The heart contracted rhythmically, and at a very reduced rate, until twenty-six hours after the administration, after which, the auricles contracted more frequently than the ventricles, and continued to do so until the heart's action ceased, seventy-three hours after the poison was injected; and, by microscopic examination of the web, it was found that a more or less feeble circulation was all this time maintained. Until this stage, no apparent change occurred in the readiness and vigour with which the striped muscles contracted when directly galvanised; their reaction continued to be alkaline, and they were perfectly flaccid. Soon after the stoppage of the heart's action the blue colour, which has been already mentioned, began to disappear, and in ninety-six hours (four days) the muscles were quite pale. No stiffness was yet observable, and galvanism still induced faint contractions. Rigor mortis commenced soon after this, but its progress was extremely slow, as galvanism produced dimples at the electrodes until 110 hours. When the frog was again examined, at 129 hours after the injection of physostigma, no muscular contraction could be produced by powerful galvanism; rigor mortis was complete; and the reaction of the muscles was found to be acid. Galvanism could produce a very faint contraction of the cardiac muscle,

limited to the points of stimulation, until about the time up to which feeble indications of retained contractility could be obtained in the voluntary muscles.

Experiment VI.—(*Temperature of Laboratory between* 52° *and* 54° *F.*)

To a frog, weighing 379 grains, four grains of extract were administered in the same way as in the preceding experiment. Motor nerve-conductivity ceased in two hours and sixteen minutes, by which time the exposed muscles were found to have become blue. The cardiac action continued rhythmical, though much reduced in frequency, until twenty-seven hours and fifteen minutes, after which, the auricles alone contracted spontaneously till forty-four hours, and then, spontaneous cardiac action entirely ceased. During all this time, the muscles were flaccid, contracted vigorously on the application of weak galvanism, and had an alkaline reaction and a blue colour. Soon afterwards, they became paler and slightly stiff, but it was not until seventy hours after the administration of the poison that galvanic stimulation failed to produce any contraction; and then rigor mortis, with an acid reaction of the muscles, set in.

These three experiments distinctly prove the absence of any paralysing effect by physostigma acting through the blood on striped muscle.

Rigor mortis is delayed for an unusual period after apparent death in cold-blooded animals, and its appearance, in mammals and birds, is certainly not hastened. In both classes this change in the condition of muscles is only indirectly affected by this substance, and that through its influence on the cardiac contractions. When the blood supply of the muscles is stopped their function is suspended, and rigidity follows; but the resulting rigor does not seem to be due, in any other than this indirect method, to the action of physostigma.

This may be more clearly demonstrated by detailing one of many experiments in which a portion of the frog was protected from the influence of the poison.

Experiment VII.—(*Temperature of Laboratory between* 52° *and* 54° *F.*)

The right iliac artery was exposed, by removing a portion of the sacrum, and tied in a frog, weighing 878 grains. Two minutes afterwards, three grains of extract, suspended in ten minims of distilled water, were injected into the subcutaneous cellular tissue at the right shoulder. In a few minutes, a condition of general paralysis existed, and shortly afterwards the skin of the tied limb was much paler than that elsewhere, this contrast becoming more marked as the experiment advanced. In an hour and twenty minutes, the sciatic nerves being exposed, it was found that the left was completely paralysed; while galvanism, applied to the right nerve, or that of the limb protected from the action of physostigma, produced active muscular contractions. The muscles of the tied limb were pale as contrasted with those to which the poison had access, and the latter were distinctly blue in colour. The non-poisoned muscles continued active until forty hours; but when examined at forty-nine hours they were acid and stiff, and did not contract when galvanised. In the poisoned parts, the functions of the motor nerves were destroyed in three hours and ten minutes; the non-poisoned, or right, sciatic continued active until thirty-two hours. It was possible to distinguish the heart's impulse on the thoracic walls, and to determine the frequency of its contractions. During the three days that immediately followed the poisoning, these steadily continued at a rate varying from seventeen to twenty-one; and on exposure, at the end of that time, fifteen feeble beats per minute were occurring. Soon after, the usual irregularities were observed, but the circulation was maintained until eighty-two hours after the injection of physostigma, as the microscope demonstrated. During all this period, the muscles everywhere, except in the tied limb, were flaccid, blue and of alkaline reac-

tion, and contracted, though latterly with diminished vigour, when galvanised: those of the tied limb were now putrefying. In 100 hours, the poisoned muscles had lost much of their blue colour, and contracted slowly and partially. In 120 hours, they were slightly stiff, and galvanism produced merely a slow surface depression at each electrode, which continued for a short time after their removal, and gradually disappeared. Such contraction could still be obtained 124 hours, or more than five days, after the injection of the Calabar bean. A similar dimpling could be produced on the heart till nearly the same time, and, therefore, long after it had lost its power of spontaneous contraction.

This experiment affords a very simple means of comparing the effect of mere absence of blood supply with the action of Calabar bean on the irritability of striped muscle. In the former case, irritability was destroyed, and the rigor of death initiated, at some time between forty and forty-nine hours. In the latter, subjection of the muscles during eighty-two hours to the influence of a blood stream conveying Calabar bean was not attended with the slightest injury; their irritability disappeared, however, forty-four hours after the circulation had been stopped by the action of this substance on the heart. Muscular paralysis was in both cases due to stasis of the circulation. It has, however, rarely happened that this irritability has continued so long as five days, in fatal cases of physostigma poisoning. The special circumstances for its production appear to be lowness of temperature and protracted continuance of the cardiac action.

These results may be seen with greater clearness if we tabulate the more important points of the above experiments, and of a few others.

Table of the Periods at which Muscular Contractility was lost, after the administration of Physostigma, and after Blood-stasis by Ligature of Vessels.

Experiment.	Weight of Frog and of Dose, in grains.	In Parts to which the Poison had Access.			In Parts cut off by Ligature.	
		Time when Circulation stopped.	Time when Muscular Contractility was lost.	Interval between stoppage of Circulation and loss of Muscular Contractility.	Nature of Operation.	Interval between local stoppage of Circulation and loss of Muscular Contractility.
V.	473 to 3	73 h.	110 h.	37 h.		
VI.	379 „ 4	28 h.	68 h.	40 h.		
VII.	878 „ 3	82 h.	124 h.	44 h.	Left iliac artery was tied.	In less than 49 h., and in more than 38 h.
VIII.	490 „ 3	52 h.	94 h.	42 h.		
IX.	292 „ 2	21 h.	75 h.	54 h.		
X.	212 „ 1·5	48 h.	104 h.	56 h.		
XI.	400 „ 2	34 h. (nearly)	74 h.	40 h. (about)		
XII.	460 „ 2	80 h.	120 h.	40 h.		
XIII.	620 „ 2	45 h.	88 h.	43 h.	Left iliac artery was tied.	44 h. 8 m.
XIV.	690 „ 3	50 h. 6 m.	72 h.	22 h. (nearly)	Left iliac artery was tied.	In less than 24 h., and in more than 19 h.
XV.	620 „ 5	0 h. 34 m.	33 h.	32 h. 26 m.	Right iliac artery was tied.	In less than 33 h., and in more than 29 h.

After this evidence, it is almost superfluous to remark that physostigma-paralysis cannot be caused by an action on striped muscle, as at least one observer of note has maintained.* Although the effects on muscle have been the first considered, it may be proper to remark here that idio-muscular irritability is the last vital property to disappear in death by Calabar bean, especially as this result has been foreshadowed in several of the experiments already given. Its loss is, moreover, only indirectly caused by physostigma, and the evidence is sufficient to show that it follows the cessation of the blood supply, which is necessary for its manifestation, in tolerably definite periods. As the circulation sometimes ceases more abruptly than at others, so does the loss of muscular irritability sometimes occur more quickly, and without much previous gradual diminution of activity. In the former case, rigor is well marked and comparatively prolonged; in the latter, and chiefly with frogs, it is slight, and the very partial nutritive activity which has been for many hours maintained by a sluggish blood-stream, favours the almost immediate occurrence of decomposition, and, therefore, of a short period of rigor mortis, when the circulation has finally stopped.

In the experiments with mammals and birds, an early and constant symptom was the occurrence of successive muscular contractions of a non-co-ordinate character; and this formed a striking contrast with the flaccid and motionless condition of the muscles which persisted throughout the poisoning of cold-blooded animals. Generally speaking, these contractions were very feeble, and consisted of slight spasmodic twitches, which, in mammals, usually began at the neck and then extended over the body, and which at first involved detached portions of the panniculus carnosus muscle only, and then apparently every muscle of the body and extremities. In the slighter cases, and, I think, where a small dose was being but slowly absorbed, a mere tremulous movement was caused of the head, body and extremities, similar probably to the "tressaillements" which CLAUDE BERNARD describes as occurring during curare poisoning,† and which have likewise been noticed with that substance by WATTERTON, and by MARTIN-MAGRON and BUISSON.‡ In one or two of my experiments, however, this muscular action became so strong, that the animal appeared as if under the action of a poison which produces convulsions. The twitches always became more marked when the poisonous effects were fully developed, they gradually diminished in strength as death approached, and they continued in a slight form for many minutes after it. Exposure of the muscles to the air, and irritation with a knife, during the autopsy, increased their strength, and even originated them in muscles and parts

* NUNNELEY on the Calabar Bean, &c. Lancet, 1863, p. 23, and Pamphlet.

† Leçons sur les Effets des Substances Toxiques et Médicamenteuses, 1857, p. 268.

‡ Action comparée de l'éxtrait de Noix Vomique et du Curare sur l'économie animale. Journal de la Physiologie de l'Homme et des Animaux, tome troisième p. 327, &c.

of muscles from which they had disappeared; and it was then observed that the whole of a muscle seldom twitched at once, but portions of it separately and in succession. I frequently removed a muscle from the dead body, and found that these twitches still continued. In one experiment, where the sartorius had been cut out of a dog, these spasms rapidly followed each other in separate portions of its substance, during ten minutes. Their duration after death varied greatly. Occasionally they ceased before the motor nerves had lost their function, while they frequently continued after their paralysis. The latter effect was well shown in an experiment where seven grains of extract were given to a large dog, by injection into a jugular vein: death, with stoppage of the heart's action and of the respiratory movements, took place in eleven minutes; in twenty-five minutes afterwards, the sciatic nerves were paralysed; and these extraordinary muscular twitches continued for other twenty minutes, or for forty-five minutes after the death of the animal. The central nervous system has no influence in causing or originating these quiverings, for division of a sciatic nerve, before the exhibition of physostigma, did not appear to impede their production. I believe the effect is due to the contact of the poison with the muscular substance itself; and this view is supported by the above facts, and by the circumstance that when a ligature was drawn round the posterior extremity of a rabbit, taking care to exclude the sciatic nerve, the muscles of that limb remained unaffected, after the administration of Calabar bean, while those of the body and of the other extremities were twitching in the usual manner.

Action on the Cerebrum.

A condition of retained consciousness with marked paralysis opposes the idea of the latter symptom being due to coma. Professor Christison has admirably described the coincidence, in his own person, of retained mental vigour with inability for movement. To distinctly prove the absence of any cerebral explanation for this paralysis, a simple experiment was undertaken.

Experiment XVI.

The brain was removed with care from a large frog, and, sometime after this, the animal was found jumping about vigorously.* Two grains of extract, suspended in fifteen minims of

* It appears somewhat startling to assert that complicated movements, of an apparently voluntary character, may continue in frogs after the removal of the brain. I first observed this in an experiment in which the spinal cord had been divided at the base of the skull; and, in describing the condition of the frog in which it was seen, I added a qualifying note ascribing the circumstance to incomplete division of the medulla. Since then, I have occasionally observed the same phenomenon; and the present experiment is conclusive in showing that some of those functions which we are in the habit of ascribing to the cerebral lobes alone, are, in frogs at any rate, shared in by the spinal cord.

May 1867.—Dr Norris enters into this anomaly, and confirms its occurrence, in his admirable paper on Muscular Irritability, in the "Journal of Anatomy and Physiology," No. 2, p. 221, *et seq.* He also refers to Lewes (Physiology of Common Life, vol ii.) as having first prominently announced this curious exception to the generally received views on nerve physiology.

distilled water, were injected into the lower portion of the abdominal cavity. It continued to jump about for four minutes; in five minutes, the first indications of paralysis occurred, at the anterior extremities; and in twenty-five, it was lying flaccid on its belly and chest, without any respiratory movements, but with retained reflex action. The sympoms advanced in their usual order until the complete death of the frog.

An action on the cerebrum cannot, therefore, be the cause of the paralysed condition which is produced. The cerebrum may, notwithstanding, be acted on by physostigma; and the results of several experiments, in which I took various quantities of this substance, appear to favour such an opinion, though, until further investigation, I cannot maintain that the effects produced were not mainly dependent on those perturbations of the circulation which are caused by this poison.

Action on the Spinal Nerves.

1. *Motor or Efferent Nerve-Fibres.*

Experiment XVII.

I injected into the jugular vein of a very large retriever dog, seven grains of extract, suspended in twenty-five minims of distilled water. In three minutes, respiration became gasping; in five, the usual twitching affection of the muscles commenced; and in ten, the dog was lying in a powerless condition. In twenty minutes, respiratory and cardiac action had completely ceased. The right sciatic nerve was then exposed, and galvanic stimulation of its trunk produced vigorous movements confined to the leg whose nerve was stimulated. The same result was obtained when the left sciatic nerve was exposed and stimulated. Both retained their motor conductivity for eight minutes after respiration had ceased; and, for some time longer, the muscles responded to direct galvanic stimulation.

Experiment XVIII.

Five grains of extract were mixed with thirty minims of distilled water, and injected into the abdominal cavity of a large and healthy female cat. Trembling occurred in four minutes, when the cat ran a short distance and fell on her face, after which she lay in any position, flaccid and unresisting. Respiration ceased in ten minutes, but the peculiar twitching of the muscles continued for several minutes longer. The heart was then exposed, and found motionless and full. On galvanising the sciatics, or otherwise stimulating them, the muscles of the posterior extremities contracted vigorously; but no evidence was obtained of the reflex activity of the cord. The sciatic nerves continued active until forty-two minutes after the death of the animal (fifty-two after the administration of the poison); but, when they were galvanised at forty-seven minutes after death, no muscular contraction was produced. The phrenic and brachial nerves also continued active for about the same time.

Experiment XIX.

Three grains of extract, suspended in eleven minims of distilled water, were injected into the subcutaneous cellular tissue in the flank of a full-grown rabbit. Trembling occurred in two minutes, and this continued, with varying strength, until seven minutes; two minutes before which respiration had ceased. The sciatic, phrenic and brachial nerves were galvanised, and found to be active; and such stimulation of the sciatics continued to produce muscular contraction until nine minutes after the cardiac and the respiratory movements had ceased.

Experiment XX.

There was injected into the subcutaneous tissue at the right flank of a rabbit, weighing two pounds, half a grain of extract in fifteen minims of distilled water. The usual tremors rapidly supervened, and in forty-five minutes the animal was dead. During the following thirty-one minutes, galvanism of either sciatic nerve produced contractions of the limb which it supplied.

Experiment XXI.

Half a grain of extract was placed in the mouth of a pigeon. In ten minutes, a profuse flow of saliva and of tears, with occasional passage of fæces, occurred; after which time the bird lay in a helpless condition, with now and then a sudden starting movement. In twenty-four minutes, its respirations had completely ceased. Until eight minutes after death, galvanism of the left sciatic nerve continued to produce movements in the left leg; but in thirteen, the nerve was quite paralysed.

Such experiments were frequently repeated; and in no case did I find that the motor nerves were paralysed before the respiratory movements had ceased, although it has occurred that they have almost immediately afterwards been so. I have found that the interval during which they remain active varies greatly in different animals, and in the same animal according to the dose of poison administered; and I believe that in the latter case the variation is in an inverse ratio. In the rabbit, motor conductivity may be retained for periods ranging from a very few (two or four) to thirty-one minutes. We cannot, therefore, account for either the condition of general paralysis or the cessation of the respiratory movements, which two form the most prominent of the symptoms of physostigma poisoning in warm-blooded animals, by an action on the motor nerves. Experiments XVII., XVIII., XIX., XX. and XXI. are sufficient to prove this. The evidence obtained by experiments with frogs is even more unmistakable. Complete destruction of all the vital functions in this animal never occurred for many hours. In animals of a higher type, the implication of one system so rapidly influences the others that it is often difficult to discriminate between the effects which are caused by the poison and those which are induced as results of the primary action. In the frog, on the other hand, the symptoms advance so slowly from one system to another that it is possible to determine distinctly the sequence of the phenomena which are due to the direct influence of physostigma.

Experiment XXII.

Five grains of extract, suspended in a few minims of distilled water, were injected under the skin over the back of a frog weighing 490 grains. A small quantity escaped in the somewhat vigorous movements which occurred when the frog was liberated. The thoracic respiratory movements ceased in ten minutes, and those of the throat in other four. Twenty-two minutes after the injection, the animal lay on its abdomen in a perfectly flaccid condition; its heart was acting feebly, at the rate of seven per minute; and pinching of the skin anywhere caused but very weak reflected movements. In one hour, no evidence could be obtained by galvanism, or by

any usual irritant, of continuance of the diastaltic function of the cord; but the heart was now contracting nine times in the minute. The sciatic nerves were then exposed; and, on galvanising either of them, movements occurred in, and were confined to, the muscles of the limb whose nerve was so stimulated. Both nerves continued to give this evidence of the conductivity of their motor fibres as long as one hour and thirty-eight minutes after the injection of Calabar bean, or one hour and twenty-four minutes after all respiratory movements had ceased. Shortly afterwards they were found paralysed.

Experiment XXIII.

Into the lower portion of the abdomen of a frog, which weighed 620 grains, a mixture of two grains of extract with ten minims of distilled water was injected. In twelve minutes, the frog was lying in a flaccid condition, and respiration had ceased. In two hours, no reflex movement could be excited. The right sciatic nerve was exposed; and its motor conductivity was found to remain. Galvanism, applied to either sciatic nerve, produced muscular contractions in the limb to which the nerve was distributed, until, but not later than, three hours and twenty-one minutes after the administration of the poison, or three hours and nine minutes after respiration had ceased, and until more than one hour and twenty-one minutes after apparent destruction of the reflex function of the spinal cord. In this experiment, it is important to note, the cardiac action was not greatly affected for more than three hours, as the dose of poison administered was comparatively small.

Experiment XXIV.

Six grains of extract, suspended in fifteen minims of distilled water, were injected into the abdominal cavity of a frog, weighing 350 grains. In eight minutes, no cardiac impulse could be discovered; and the heart was then exposed and found motionless, dark and flaccid. Respiratory movements ceased in eight minutes. It was determined, on irritating the skin with sulphuric acid, that reflex movements could not be obtained two hours and a-half after the injection; but they were produced until nearly this time. *The motor conductivity of the sciatic nerves was retained for twenty-nine hours.*

It thus appears that the motor nerves always remain active after the co-ordinated movements of respiration have ceased, and after the condition of complete and flaccid paralysis has existed for long. On this point, therefore, I cannot agree with Harley, who considers that physostigma is a respiratory poison only, and that the early production of asphyxia is caused by paralysis of the motor nerves.*

The protracted interval in the last experiment, between the administration of the poison and the loss of motor conductivity, must be looked upon as a very exceptional one, and as due to the poison having so quickly paralysed the heart that the usual phenomena were not produced. In Experiments XVII., XIX., XXI., XXII., and XXIII., the motor nerves appear to have lost their function sooner than naturally happens in death from cardiac paralysis or from asphyxia. But that they really did so must have remained a mere impression, had it not been that we can, in frogs at any rate, definitely prove a special action on the

* Journal de l'Anatomie et de la Physiologie, 1864, p. 141, *et seq.*; and British Medical Journal, Sept. 3, 1863.

motor nerves, by protecting a portion of the animal from the poison. Such a proceeding cannot be avoided by determining the interval which naturally elapses between the moment of death and the time at which loss of function occurs in these nerves. This interval varies greatly in different classes of animals, and also in different individuals of the same species; and, hence, the data which have as yet been accumulated on this point are not available for such purposes as the present.

Experiment XXV.

The sacrum was excised from an average-sized frog, and, in that manner, the lumbar nerves and the abdominal aorta were exposed. The aorta was ligatured above its bifurcation into the two iliacs, and, immediately afterwards, an average poisonous dose of extract was placed in the frog's mouth. In ten minutes, respiration had ceased; but the frog continued jumping about for other four minutes, when it quietly and gradually subsided on its abdomen and chin. Reflex movements could be excited by irritating the skin anywhere until one hour and seven minutes after the administration; but, for some time before this, a great increase in the strength of the stimulant was required. The skin of the posterior extremities had now become paler, while that of the body, anterior to the ligature, had assumed a much darker colour than it had before the experiment; and the exposed heart was found beating twenty per minute, with regularity and in proper rhythm. The brachial nerve was then laid bare, and was found active; *but this condition ceased two hours and eleven minutes from the commencement of the experiment, and both brachials were then perfectly paralysed.* At this time, weak galvanism applied with closely approximated poles to sections of the spinal cord produced no effect; but when the lumbar nerves below the ligature were galvanised they caused vigorous contractions of the posterior extremities. *On the following morning, this motor nerve activity still remained, in the parts protected from the poison:* but the contractions were now feeble, as muscular rigor was commencing below the ligature; and, in a few hours longer, the muscles separated from the circulation ceased to contract. The muscles of the poisoned portion, in which the motor nerves had been long paralysed, were still quite flaccid, alkaline and irritable; and rigor mortis did not occur in them until the following day.

Experiment XXVI.

The right iliac artery and the right ischiadic vein were tied in a frog, weighing 620 grains; and five grains of alcoholic extract of physostigma, suspended in thirty minims of distilled water, were injected into the abdominal cavity. In twenty minutes, voluntary movements had completely ceased, there were no respirations, and the frog lay in a perfectly flaccid condition. Fifty minutes after the administration of the poison, the left sciatic nerve was exposed. Very weak galvanism of the nerve-trunk caused contractions of the limb; and continued to do so, on occasional observations, till two hours and ten minutes from the commencement of the experiment, or till fifty minutes after the respirations had ceased. In other fifteen minutes, however, the nerve was found to be completely paralysed. The right sciatic nerve, which had been protected from the influence of the poison, by ligature of the blood-vessels of the limb, was examined in a similar manner. *Its motor conductivity continued unimpaired for at least five hours longer than that of the poisoned nerve.*

These are examples of numerous experiments which were undertaken for the special purpose of determining whether Calabar bean has any action on the spinal nerves. They prove undoubtedly that it has the power of destroying their motor conductivity; but it has also been demonstrated that this is not the cause of

paralysis and death in mammals, or of complete loss of voluntary power in frogs. The next question which suggests itself for solution is the somewhat interesting one of the portion of nerve acted on; for, in the case of more than one toxic substance, it has been determined that loss of motor nerve-function does not of necessity imply that both the periphery and nerve-trunk have been affected.

Experiment XXVII.

Immediately after ligature of the left ischiadic artery and vein, two grains of extract were placed in a subcutaneous cavity at the back of an active large frog. Before twenty minutes, respiratory movements had ceased; while the heart was then acting rhythmically, at the rate of thirty beats per minute. One hour after the administration of the poison, the right (or poisoned) sciatic nerve was exposed, and found active; but in other twenty minutes, strong galvanism applied to any portion of its trunk could not produce contractions in the muscles to which it was distributed, and an examination of the brachial nerves proved them also to be paralysed. The left sciatic nerve was, however, perfectly active. When it was galvanised, movements, confined to that limb, were produced in the muscles below (or distal from) the ligatures. *These ligatures were on the thigh; but stimulation of the nerve above them, or of the lumbar nerves on the same side, was followed by energetic muscular contractions below the points of ligature.*

The muscles were everywhere active, and continued so for several days; and those of the non-poisoned limb were the first to pass into rigor mortis. The sciatic nerve of the non-poisoned limb, and the lumbar nerves of the same side, continued active for many hours; but their loss of function occurred several hours before that of the muscles below the ligatures.

This evidence, which has been frequently confirmed, is in favour of the view that the motor paralysis caused by physostigma is due to an action on the nerve endorgans, or peripheral terminations, and not to one on the trunk. We may obtain even more distinct proof, by a slight modification of this experiment.

Experiment XXVIII.

An incision was made down the centre line in the right posterior extremity, from the back of the knee to the ankle, of a frog, weighing 876 grains; and in this way the gastrocnemius muscle was completely exposed. It was carefully dissected from its connections, excepting that its origin and insertion, and the nerve fibres entering it, were untouched. All its blood-vessels were ligatured, and the cut through the skin was closed by sutures.

Immediately after the above operative procedure, three grains of extract, in fifteen minims of water, were injected under the skin of the back. Reflex movement could not be excited an hour and five minutes afterwards, while the heart still continued to contract. Both sciatic nerves were then exposed. Galvanism of the left produced no contraction; while galvanism of the right caused energetic movements of the limb, which, moreover, did not extend to the toes. Five hours after the administration of the poison, this condition continuing, the left gastrocnemius was exposed, and the right again laid bare by cutting the sutures. Galvanism of the right sciatic demonstrated visibly contraction of the right gastrocnemius, but of no other muscles of that limb; and no result followed stimulation of the left sciatic, although the left gastrocnemius muscle contracted vigorously when the poles were applied to its surface.

It is thus shown that some of the endorgans of a motor nerve may have their conductivity destroyed while others remain active during the retained

vitality of the nerve trunk, and that this contrast in condition depends on the access or not of the poison—a clear demonstration of the power of physostigma to paralyse the nerve terminations. This action has been hitherto overlooked.

Calabar bean is, therefore, now added to that very limited class of neurotic agents which affect the motor endorgans. Indeed, only two substances, as far as I am aware, were previously known to possess this remarkable action. For a considerable time after the brilliant, and perhaps unequalled, researches of CLAUDE BERNARD,* curare stood alone as such a substance: when KÖLLIKER discovered that conia has a similar action;† and his observations have been recently confirmed by GUTTMANN.‡

Physostigma is, however, peculiar in the method in which it so acts; a very prolonged contact with the nerve terminations, and a long continued circulation of poison-bearing blood, being apparently necessary. In warm-blooded animals, this paralysis of the motor endorgans may, therefore, be easily overlooked; but in frogs, with localised poisoning, it is conspicuously displayed, as in the experiments which have been given.

It is interesting to remark the different conditions which are produced in the functional vitality of nerves and muscles when physostigma is administered to a frog after the vessels of one of its limbs have been ligatured.

Experiment XXIX.

Immediately after ligaturing the right ischiadic artery and vein of a frog, which weighed 609 grains, two grains of extract, in fifteen minims of distilled water, were injected into the subcutaneous tissue of the back.

One hour and twenty-five minutes afterwards, the heart was found beating seventeen times per minute.

The two gastrocnemii muscles, with their femur attachments, and a portion of each of these bones, along with the sciatic nerves from their terminations in the gastrocnemii to the lumbar plexus, were then removed. These parts were so arranged, that an interrupted current from one DANIELL'S cell and DU BOIS REYMOND'S induction apparatus, could be transmitted simultaneously through either both nerve trunks, or both muscles, by the turn of a key.

a. Examination of the Nerves (one hour and fifty-four minutes after the administration of the poison).—The galvanic current was first passed through the sciatic nerves. Distinct tetanus of the non-poisoned muscle was caused when the secondary coil stood at 63° on the scale; the poisoned was at perfect rest. The current was gradually strengthened by advancing the secondary coil; when this reached 53°, but not before, the poisoned muscle was thrown into tetanus.

b. Examination of the Muscles.—Immediately afterwards, the current was passed directly through both muscles. The poisoned gastrocnemius contracted when the secondary coil reached 63°; the non-poisoned did not do so until this was advanced to between 52° and 53°: that is to say, the poisoned muscle was thrown into tetanus by a weaker current than was required to produce the same effect in the non-poisoned muscle.

* Leçons sur les Sub. Tox. &c., 1857, pp. 238–413.

† Verh. d. phys.-med. Ges. zu Würzburg, 1859, vol. ix., part 2, p. 55, *et seq.*; Virchow's Archiv. x., p. 235; and other papers.

‡ Berliner Klin. Wochenschr., No. 5–6, 1866. Quoted in RUTHERFORD'S Report on Physiology; Journal of Anatomy and Physiology, No. 1, 1866, p. 155.

Experiment XXX.

A similar experiment was performed on another frog, also poisoned with two grains of extract, but weighing only 464 grains.

a. Examination of the Nerves (two hours and twenty minutes after the injection).—The galvanic current was passed along portions of both sciatic nerves simultaneously. When the secondary coil reached 52°, the non-poisoned gastrocnemius was thrown into tetanus, the poisoned remaining inactive. It was then slowly advanced; and at 30°, faint contractions occurred in the poisoned gastrocnemius, of a partial character, as if only a few muscular bundles, and not the whole muscle, were contracting, and which continued for a few seconds only, and did not recur although the secondary coil was advanced, after an interval for rest, to 0° (the strongest current from this arrangement). The stimulus was then applied to the trunk of the non-poisoned nerve above the position of the ligatures, and, therefore, where it must have been in contact with the poison; contraction again occurred when the secondary coil was at 52°.

b. Examination of the Muscles.—When the secondary coil reached 55°, tetanus was produced in the poisoned gastrocnemius; and at 54°·5, the non-poisoned muscle was thrown into tetanus.*

It would thus appear that motor nerve excitability or conductivity is diminished and then destroyed by physostigma (this change being produced at the endorgans), while retained for a long time thereafter in those parts of the same animal which have been guarded from the access of the poison. It is also seen that the effect on idio-muscular contractility is exactly converse; that property being uninjured by the mere presence of physostigma, while diminished and destroyed by stoppage of the circulation.

It is usually asserted that division of a nerve, previous to the exhibition of any substance that affects its vitality, is a sufficient method for determining the position of its primary implication, and, therefore, sufficient for determining the direction in which this extends. This proceeds upon the supposition that when it is the nerve trunk near its origin that is first affected, extension of the poisoned condition to the distal portion will be delayed by intermediate division of the nerve. BERNARD attempts in this way to prove that the primary paralysis of the motor nerve endorgans by curare extends from them, along the trunk of the nerve, towards the cord.† The paralysis of the motor nerves after strychnia is said to proceed in a direction exactly the reverse,—from the origin to the periphery.‡ It appeared of some interest to examine this question with Calabar bean, for with it we would not expect that previous division of the trunk should delay the implication of the nerve endorgans, as this precedes the paralysis of the trunk.

Experiment XXXI.

I exposed the two sciatics of a frog for a short distance, and both equally, and divided the left nerve with a very sharp pair of scissors. A fatal dose of Calabar bean was then administered. When reflex movement could no longer be excited (one hour and two minutes after the administration), the right sciatic was galvanised; but no contractions were caused. The left was stimulated, at the cut extremity of its distal portion, with the same current, and active movements of the left leg and toes followed. The galvanism was repeated, at intervals of five minutes,

* In Experiments XXIX. and XXX. I have thankfully to acknowledge the valuable assistance I obtained from Dr RUTHERFORD.

† *Loc. cit.*, p. 312. ‡ *Ibid.*

for other twenty minutes; with the same negative result in the case of the uncut nerve, and with continuation of activity in the cut one. The energy of the contractions then gradually diminished; but the distal portion of the cut nerve was not paralysed until twenty-eight minutes after the loss of motor nerve conductivity in the portion of its trunk proximal to the spinal cord, as well as in all the other nerves of the body.

Experiment XXXII.

In a second experiment, the distal portion of the divided nerve retained its conductivity forty-five minutes longer than the undivided nerve.

Another experiment contains some further results, which are worthy of being shortly mentioned.

Experiment XXXIII.—(*Temperature of Laboratory*, 56° *F.*)

Performed in August 1866.

I exposed a small and equal portion of each sciatic nerve in a frog weighing 515 grains, and cut through the right nerve-trunk. One grain and a-half of extract was then injected into the cellular tissue under the skin of the back.

In one hour and thirty minutes, no variety of stimulation could excite reflex movements: and when the left, or uncut, sciatic was then gently galvanised, faint contractions of the left toes were all that was produced; while the same interrupted current caused vigorous contractions when applied to the distal portion of the cut nerve. This condition continued, the contractions produced by the left (uncut) nerve becoming gradually fainter, until three hours and twenty-two minutes after the poison had been injected, when this nerve became perfectly paralysed, as well as all the other motor nerve fibres and the proximal portion of the cut sciatic. The distal portion of the cut (right) sciatic seemed all this time quite unaffected; and when it was galvanised, the muscles with which it was connected contracted with vigour. This condition lasted for fifteen minutes, *when galvanism of the uncut nerve again produced a faint twitch of several of the left toes.* By-and-by, it recovered all its former activity, and the return to vitality was shared in by the other temporarily paralysed nerves. On the following morning the frog was perfectly well and jumping about.

From such data it cannot be concluded that the motor nerve fibres are paralysed by a centripetal progression of the poison. There seems only to be a connection between the rapidity of paralysing effect, on the one hand, and the subdivision of nerve substance with facility of contact of poison, on the other; as the motor trunks were undoubtedly affected when it was impossible that they should be influenced by an extension of the poisonous action from their endorgans. I cannot advance any very satisfactory explanation of this delay in the action. Probably it is caused by the irritation of the vaso-motor nerves in the trunk of the sciatic, which the section of the latter at first produces, and which is followed by contraction of the capillaries in the parts supplied by them.* During this contraction, a much smaller quantity of poison-bearing blood is brought in contact with the endorgans of the cut than with those of the uncut nerve, and, as a very prolonged contact of the poison appears necessary, this may be sufficient to account for the delay; while the recovery from the irritation of the section, which soon

* Such contraction has been directly demonstrated by LISTER and others after division of the sciatic nerve. See "An Inquiry regarding the parts of the Nervous System which regulate the contraction of the Arteries;" Philosophical Transactions, 1858, vol. cxlviii., p. 607.

occurs, again allows of a free circulation and of consequent paralysis of the motor endorgans of the divided nerve.

2. *Afferent Nerve Fibres.*

The discussion of the influence that those fibres in the spinal nerves that conduct impressions to the cord exert in producing paralysis, will be a short one, as it can be readily shown that their effect is negative. It will be sufficient to notice that in mammalians it was always possible to obtain evidence of their activity as long as the functions of the spinal cord were retained, and that, therefore, they were in no wise concerned in the production of the general flaccidity and loss of motor power which is caused by Calabar bean. The same evidence was obtained in frogs, and could in them be distinctly shown by localization of the poisoning. In place of the function of these nerves being lessened, I believe that it is generally increased, so that movements may be excited more readily after the action of the poison than before it.

Experiment XXXIV.

I tied the right ischiadic artery and veins of a frog weighing 573 grains, and suspended it by the lower maxilla. Soon after, a silk thread was drawn over various parts of the skin, including the right leg, without exciting any reflex movement. On dipping the feet, separately, into dilute sulphuric acid (five minims of oil of vitriol to twelve ounces of water) reflex movements occurred with each, after 80 beats of a métronome set at 100 in the minute. I then injected into the abdomen two and a-half grains of extract in fifteen minims of water. In one hour and three minutes, on the left foot being dipped into the acid, reflex movements occurred in the right in 190 beats of the métronome; but no movement followed in the left limb in 200 beats, nor when the poisoned foot was placed in stronger acid (10 min. to 12 oz.), while this caused energetic contractions of the non-poisoned portion of the right limb. The reflex activity of the spinal cord was, therefore, very greatly diminished, and still the afferent nerves continued active. The silk thread which was formerly employed was now drawn over the skin of the right leg below the ligatures, and, as before, it produced no diastaltic movement. *On applying it, however, to the skin of the left leg and of the other poisoned regions, twitches constantly occurred in the toes of the right leg, and only occasionally, and of a very feeble character, in the poisoned region.* This was repeated, at intervals, during the next ten minutes, with the same result. After this, the thread ceased to excite diastaltic movements; but the spinal cord had now lost its vitality, and no movement could be produced even when it was directly galvanised.

The afferent nerve fibres, in this experiment, retained their original activity longer than the efferent, and at least up to the time at which they could not be tested, because of the loss of the diastaltic function of the spinal cord. The increase of excitability in the afferent fibres, where these had been acted on by blood conveying physostigma, cannot be due, in the slightest degree, to any spinal cause, for the effect did not occur in the right leg, to which access of the poison had been cut off by ligature. Besides, measurement of the rapidity with which reflex movements followed the application of a stimulus to either poisoned or non-poisoned parts gave such proof of marked depreciation having occurred before the afferent excitability had been increased as is sufficient of itself to eliminate any spinal influence.

ACTION ON THE SPINAL CORD.

Having now excluded the encephalon, muscles and spinal nerves, we are led to conclude that the production of the paralysis by physostigma is due to an action on the spinal cord. Such an opinion was originally expressed by myself after a careful general consideration of the symptoms, but no subsequent investigator has coincided with my theory except LASCHKEWICH,* from whose excellent paper I have derived more than one hint for the further and special examination of the spinal effects of Calabar bean.

I quote the following experiment from my previous paper on this subject, as it affords an excellent example of those general paralytic symptoms that first induced me to refer the principal neurotic action of physostigma to the spinal cord.

Experiment XXXV.

"Five and a-half grains of the fine powder of the kernel were made into pills, and swallowed by a buck rabbit, eight months old.

"A slight degree of paralysis was seen in the posterior regions, in ten minutes, and, soon after, they yielded, the anterior portion of the trunk remaining supported by the fore-limbs. In fifteen minutes, the fore-legs gave way, and fæces were passed. In twenty minutes, the respirations became noisy, reflex action was not abolished, and the pupils contracted. In thirty minutes, the rabbit submitted to be placed in any position. In thirty-five minutes, the respirations became extremely noisy, and accompanied with muscular spasm. Fæces and urine were passed, and reflex action could not be induced by puncturing the skin. General, but slight, muscular spasms now occurred frequently; the eyelids did not contract when the eyeball was pricked, and the respiratory stertor ceased. In forty minutes, a general spasmodic contraction of the muscles occurred, and, in forty-one minutes, all respiratory movement had ceased.

"*Autopsy*, immediate. The cut muscles contracted. The heart was acting at the rate of seventy-two per minute; and this ratio gradually diminished till it ceased, thirteen minutes after death. The brain was rather darker than usual, and no change could be perceived in the spinal cord. The cerebro-spinal fluid was in abnormal abundance. The large veins were distended, and the right chambers of the heart were engorged with dark blood. . . . The vermicular action of the intestines was well marked, and all the viscera contained an abnormal excess of dark blood. The muscular system was flaccid, but contractions could be caused by irritation of the nerves."

This experiment formed one of a series undertaken to discover the smallest dose which could produce death in a full grown rabbit.† The quantity employed, five and a-half grains of the kernel, was the smallest that could do so.

As evidence of the same character, I add the following from many subsequent experiments:—

Experiment XXXVI.

One grain and a-half of extract, suspended in fifteen minims of distilled water, was injected into the abdomen of a small collie dog. The animal was rapidly affected with inability to

* VIRCHOW's Archiv. Februar, 1866. † *Op. cit.* sect. v. experiment viii.

stand, tremors, lachrymation, defæcation and urination; and in eleven minutes all respiratory movement had ceased. The spinal cord was immediately exposed, but the strongest galvanism, consistent with localisation of the current, applied to various portions of its substance, failed to excite any movement of the body. A sciatic nerve was then exposed; and slight stimulation of it produced vigorous contractions of the limb, but no reflex movement. Lastly, the thorax was opened; and the heart was found contracting thirty-two times in the minute, in perfect rhythm and with regularity, although the diastolic pause was somewhat prolonged. Thirty-nine minutes after death, the cardiac beats were ten per minute, and the sciatic and other nerves could still transmit excito-motory impressions to their muscles.

Such data are sufficient, after the former results, to prove the action of Calabar bean on the spinal cord of mammalians, as far as it is possible to do so. Where larger doses are given, the evidence is not so distinct; as, along with complete loss of reflex function, the heart is found paralysed at death; and it is well known that, in the animals in question, stoppage of the circulation is rapidly followed by loss of reflex function. Still, from the above, and from other experiments which will follow, it can be conclusively proved that physostigma has a special and primary action on the cord.

Experiment XXXVII.

Performed December 1866.

After tying the left femoral artery and vein of a frog, weighing 430 grains, I injected two grains and a-half of extract into the cellular tissue of the back. In an hour and twenty-two minutes, the reflex function of the cord, as tested by stimulation of the skin by galvanism and by sulphuric acid, was completely destroyed; but the exposed heart was found acting regularly and rhythmically, though only at the rate of twelve beats per minute. The two gastrocnemii muscles, with their attached sciatic nerves, and the portions of femur into which these muscles are affixed, were then removed. The poisoned nerve and muscle were arranged in the usual manner on Du Bois Reymond's modification of Helmholtz's myographion—an apparatus designed to measure the rates of conduction along nerve fibres. The curved lines produced by stimulation of two portions of the nerve, differing in length by one inch and a-half, were found to correspond so exactly that the period during which the impression travelled over the one inch and a-half of poisoned nerve could not be measured. The non-poisoned nerve gave the same result.

This experiment was undertaken to determine whether physostigma gradually lessens the rate of conduction in motor nerves, as curare is stated to do.* It was worthless to answer this question, as was also another immediately afterwards performed with the same result; for the frogs employed were in too irritable a condition: but its value is evident in considering the action of physostigma on the spinal cord. *For the diastaltic function of the spinal cord was completely destroyed, while the poisoned and non-poisoned motor nerves were in so equally active a condition, that the difference between the times in which impressions travelled along two portions of the same nerve, differing in length by one inch and a-half, could not be measured in either, even by a delicate instrument specially adapted for this purpose.*

* A. von Bezold; Monats Bericht der Berlin: Akad. 1859.

In the further investigation of the effect on reflex movements, there is no process which yields so conclusive results as that in which frequent measurements are made of the interval that elapses between the application of a stimulant to the extremity of an afferent nerve and the resulting reflex contraction.*

Experiment XXXVIII.

The spinal cord was divided at the occiput of a frog, weighing 460 grains. It was suspended by the lower jaw, and the reflex activity tested by dipping the web of both posterior extremities into dilute sulphuric acid (ten minims oil of vitriol to twelve oz. of water). The exact time which elapsed between the contact of the foot and the resulting reflex movement was ascertained by the beats of a métronome, set at 100 in the minute. Before the administration of the poison, the reflex movement occurred in twelve beats. Two grains of extract, in water, were injected into the abdomen.

In 5	minutes, reflex movement occurred in	15	beats.
10	„ „	31	„
15	„ „	40	„
20	„ „	57	„
25	„ „	69	„
30	„ „	82	„
35	„ „	106	„
40	„ „	134	„
45	„ „	165	„
50	„ „	181	„
55	„ „	192	„
1 hour 5 min.	no reflex movement after	250	„
1 „ 15 „	strong acid caused no movement.		

The sciatic nerves were then exposed; and weak and carefully localised galvanism applied to either trunk caused energetic contractions of the limb below the portion stimulated, which could be obtained until two hours after the injection of the poison.

It seemed important to ascertain the coincident changes that take place in the heart's action; and for this purpose several experiments were undertaken, of which the following is an example. By a slight adjustment of the frog's body, the cardiac impulses are easily seen and counted.

Experiment XXXIX.

A frog, weighing 460 grains, was suspended by its lower jaw. The average of the cardiac contractions, during ten minutes, was forty-five per minute. The two feet were alternately stimulated, every five minutes, by contact with dilute sulphuric acid (10 minims of oil of vitriol to 12 oz. water); a vessel containing the acid being gently raised so that the fluid covered the whole foot. A métronome, set at 100 in the minute, was employed to determine the interval between the application of the irritant and the resulting reflex movement.

* This method of examining reflex activity seems to have been first recommended by Von Türck in 1850 (Ueber den Zustand der Sensibilität nach theilweiser Trennung des Rückenmarks); and its value has been brought more prominently into notice by Dr J. Setschenow (Physiologische Studien über die Hemmungs-mechanismen für die Reflexthätigkeit des Rückenmarks im Gehirne des Frosches, Berlin, 1863).

15 minutes before poisoning,	the right foot	was drawn up in	13 beats.
10 „ „	left „	„	11 „
5 „ „	right „	„	9 „

Two grains of extract, in 15 minims of water, were now injected into the subcutaneous tissue of the back.

In 5 minutes,	the left foot	was drawn up in	14 beats.	
10 „	right „	„	22 „	
15 „	left „	„	36 „	the heart contracted per minute, 36
20 „	right „	„	28 „	
25 „	left „	„	39 „	„ 18
30 „	right „	„	52 „	
35 „	left „	„	49 „	„ 18
40 „	right „	„	72 „	
45 „	left „	„	60 „	„ 17
50 „	right „	„	76 „	
55 „	left „	„	88 „	„ 15
1 h. & 0 min.	right „	„	99 „	During this interval, the heart's impulse could not be counted, but the frequency of its contractions was probably much lower than 15 per minute.
1 „ 5 „	left „	„	115 „	
1 „ 10 „	right „	„	111 „	
1 „ 15 „	left „	„	138 „	the heart contracted per minute, 16
1 „ 20 „	right „	„	169 „	
1 „ 25 „	left „	„	186 „	„ 20
1 „ 30 „	right „	„	192 „	
1 „ 35 „	left „	twitched slightly in	230 „	„ 20

After this, no reflex movement occurred when either foot was dipped in much stronger acid; but galvanism applied to the exposed sciatics caused vigorous movements, which occurred only in the limb whose nerve was stimulated. The cardiac contractions, in half an hour, became as frequent as twenty-five per minute; and, on the following morning, the thorax was opened, and rhythmical contractions were perceived, at the rate of twenty-four per minute. A microscopic examination was at this time made of the web, and a circulation was discovered in its capillaries.

A few experiments were undertaken in order to eliminate the possible effect of Calabar bean on the motor nerves in producing this gradual depression of reflex activity.

Experiment XL.

The femoral artery and vein in each posterior extremity of a frog, weighing 540 grains, were ligatured, and the animal suspended by the lower maxilla. Both feet were simultaneously dipped in dilute sulphuric acid (10 min. of oil of vitriol to 12 oz. of water); and the interval between the contact and the resulting reflex movement was measured, as before, by the beats of a métronome.

15 minutes before the poison was injected,	reflex movement	occurred in	10 beats.
10 „ „		„	9 „
5 „ „		„	11 „

I then injected three grains of extract, in ten minims of water, under the skin of the back.

In 10 minutes,	reflex movement occurred in	28 beats.
20 „	„	24 „
30 „	„	71 „

In 40 minutes,		reflex movement occurred in	80 beats.
50	,,		,,	101 ,,
1 hour and	5 minutes		,,	118 ,,
1 ,,	15 ,,		,,	136 ,,
1 ,,	20 ,,		,,	150 ,,
1 ,,	30 ,,		,,	200 ,,
1 ,,	40 ,,		no movement in	200 ,,

A stronger acid was substituted (20 min. to 12 oz.).

In 1 hour and	50 minutes,	reflex movement in		111 beats.
2 hours	0 ,,		,,	160 ,,
2 ,,	10 ,,		,,	173 ,,
2 ,,	20 ,,		,,	198 ,,
2 ,,	30 ,,	no reflex movement in		220 ,,
2 ,,	35 ,,	strong acid caused no reflex movement.		

The heart was exposed; and it was found contracting in proper rhythm, nineteen times per minute. The sciatic nerves were active.

On the following day, the heart was contracting twenty times per minute; and galvanism of the sciatic nerves caused feeble muscular contractions in the posterior extremities, but no diastaltic movement. The frog ultimately died.

Experiment XLI.

After ligature of its left femoral artery and vein, a frog, weighing 495 grains, was suspended as in the previous experiments. The reflex activity was tested with the following result:—

Métronome, 100 to 1 minute; acid = 5 min. to 12 oz. water.

At 20 minutes	before poisoning, the	left leg	was drawn up in	24 beats.
15 ,,	,,	right ,,	,,	19 ,,
5 ,,	,,	left ,,	,,	28 ,,

Three grains of extract, in water, were then injected subcutaneously at the back.

In 5 minutes, the	right leg	was drawn up in	24 beats.
10 ,,	left ,,	,,	30 ,,
15 ,,	right ,,	,,	36 ,,
20 ,,	left ,,	,,	56 ,,
25 ,,	right ,,	,,	72 ,,
30 ,,	left ,,	,,	90 ,,
35 ,,	right ,,	,,	150 ,,
40 ,,	no movement when either foot was kept in contact with the acid during 200 beats.		

A stronger acid was substituted, of the strength of 10 minims of oil of vitriol to 12 oz. of water.

In 45 minutes, the	right leg	was drawn up in	89 beats.
50 ,,	left ,,	,,	104 ,,
55 ,,	right ,,	,,	108 ,,
1 hour	left ,,	,,	172 ,,

A still stronger acid, 20 minims to 12 oz., was now used.

In 1 hour 5 minutes, the	right leg	was drawn up in	85 beats.
1 ,, 10 ,,	left ,,	,,	118 ,,
1 ,, 15 ,,	right ,,	,,	138 ,,
1 ,, 20 ,,	left ,,	,,	140 ,,

In 1 hour	25 minutes,	the right leg	was drawn up	in	145	beats.
1 „	30	„	left „	„	158	„
1 „	35	„	right „	„	160	„
1 „	40	„	left „	„	180	„
1 „	45	„	right „	„	197	„
1 „	50	„ no reflex movement of either after			250	„ *

Both sciatic nerves were found to be active when directly stimulated, but the contractions were confined to the limb whose nerve was galvanised.

This, and the experiment which precedes it, are conclusive in showing that the diminution, and then destruction, of the diastaltic function are not interfered with when physostigma is only prevented from acting on peripheral portions of the reflex apparatus. When, however, it is permitted to act on the nerve endorgans, but is prevented from reaching the centres, the effect is very different.

Experiment XLII.—(*Temperature of Laboratory,* 53° *F.*)

I opened the abdomen of a frog, and, with great care, tied, and, in some instances, cut through, all the blood-vessels that entered the spinal canal from the lower edge of the scapula to the coccygeal extremity of the sacrum, and then divided the spinal column at the higher of these points. By this means, the blood was only prevented from reaching a limited portion of the cord; so that, though unable to convey physostigma to the reflex centre for the posterior extremities, it could still do so to those extremities themselves, and to all other parts of the body. Voluntary movements of the limbs and of the body, anterior to the divided portion of the cord, occurred when the frog was set free; and irritation of the posterior parts was promptly followed by reflected movements confined to them. The heart, which the operation had partially exposed, was contracting sixty-eight times per minute.

A large dose of extract was administered by the mouth. In twenty minutes, only very faint reflex movements could be excited when the anterior feet were irritated, while gentle stimulation of the posterior caused energetic reflex movements; and the heart was acting at the rate of eighteen per minute. In little more than an hour, the diastaltic function was completely abolished in the anterior half of the body, while the conductivity of the brachial nerves remained; and, still, a slight pinch or weak galvanism of the posterior webs was followed by pretty active reflex movements. The heart had now stopped. *The diastaltic activity of the posterior half of the body continued for two hours after it had disappeared in the anterior.*

The mere prevention of the access of physostigma to a segment of the cord, while it was allowed to act directly on all other parts of the body, had, therefore, the effect of delaying the loss of reflex function in the parts connected with that non-poisoned segment. We have, already, frequently seen that protection of the endorgans and of portions of the nerve trunks from the poison does not delay or at all influence the gradual impairment and final destruction of the reflex function that ensue on its access to the cord. The conclusion is only logical, that Calabar bean produces a destruction of the diastaltic power by an action on the spinal cord.

It is not superfluous to observe that I do not in these experiments ignore the effect on the cord of mere stoppage of the circulation. When the heart is quickly

* In all these experiments, the parts which had been dipped in the acid solution were immediately washed with distilled water. The destructive action of the acid was thereby reduced to a minimum.

paralysed, as frequently happens, reflex movement will, on that account alone, be soon impossible. Experiments in which the heart is so affected are nearly worthless as far as the investigation of the spinal action of physostigma is concerned, and it would be needless to detail them. With the object of bringing forward as clear evidence as possible on this subject, I have selected experiments in which the cardiac action was only impeded, and in which it continued after the abolition of reflex power.

It is well known that strychnia causes an exaggeration of the reflex activity. Whether this is produced by an action on the spinal centre, or by one on the afferent nerves, is yet a question in dispute;* but it is obvious that a substance that diminishes this function at the cord, will also diminish the reflex activity consequent on the administration of strychnia, whatever be the special part of the reflex circle affected by that administration. Such a result is produced by physostigma, and therefore, adds to the many proofs of its spinal action.

Experiment XLIII.

I placed a small drop of solution of strychnia (Brit. Pharm.) on the back of a frog. This produced tetanus in four minutes; when a considerable dose of physostigma extract was inserted into the animal's mouth, the manipulations necessary for which excited a series of violent emprosthotonic spasms. Four minutes after the Calabar bean was exhibited, a decided diminution occurred in the frequency and severity of the convulsions; and, in nine minutes, they had lost their tetanic character. In forty minutes, it was difficult to excite even a faint reflex movement by pretty strong galvanism of any part of the body; and, soon after, reflex action had completely disappeared, even when the exposed sciatic nerves were galvanised.

By comparing the last experiment with one in which the strychnia effects were not interfered with, it is easy to show what part the Calabar bean played in thus interfering with the peculiar action of strychnia.

Experiment XLIV.

A frog was selected of the same weight as the last, and in every other respect as nearly resembling it as possible, and a small drop of solution of strychnia (Brit. Pharm.) was placed on its back. Tetanus occurred in four minutes; and violent convulsions of a tetanic character followed each other at intervals, and could be excited by the slightest touch, during the next six hours, after which the observations were stopped.

This evidence may be still further increased, if we produce well-marked paralysis by physostigma and then administer strychnia without causing its peculiar action.

Experiment XLV.

A frog was selected of the same weight as those employed in the two preceding experiments, and a dose of extract of Calabar bean was inserted under the skin of its back. This acted with considerable rapidity, so that, in twelve minutes, respiratory and voluntary movements had ceased, and reflex action was sluggish. A small drop of solution of strychnia (Brit. Pharm.) was then placed on a wound made through the skin of the back. Twenty minutes after this, reflex movement could still be excited; but during all this time no tetanic convulsion, nor even exaggeration

* MARSHALL HALL, BROWN-SÉQUARD, BONNEFIN, MARTIN-MAGRON and BUISSON maintain that strychnia causes tetanus by an action on the spinal cord; CLAUDE BERNARD and STANNIUS are the principal champions of the opposing theory of its action on the sensory nerves.

of reflex action, had occurred. Thirty-eight minutes after the strychnia had been applied, the diastaltic function had disappeared, though galvanism of the exposed sciatics still caused muscular contractions, and though the heart was beating at the rate of twenty-eight per minute.

The subject of opposing physiological actions has been a favourite one with many writers on poisons, and "antagonistic" effects have been largely discussed, as might be expected from their interest and practical applications. Nicotia,* aconitia† and curare‡ have been proposed as counter agents to strychnia, and atropia has been proposed as one to morphia.§ Calabar bean has been, before now, pointed out by myself and others as an opponent in action to strychnia; and, as with curare, its application to the treatment of tetanus has been recommended. I believe that no other drug so directly diminishes reflex action, and is, therefore, so likely to be employed with advantage in tetanus, as physostigma. Curare opposes spasm by paralysing motor nerves, nicotia by destroying muscular contractility; but physostigma attacks (if we may use the word) the spinal cord which is necessarily implicated as the centre of every diastaltic action. There seems to be no reason why it should not always prove a certain cure in traumatic tetanus. Its success in strychnia poisoning will probably depend on the quantity, in relation to the case, that has been administered; as this poison may be considered to have two fatal doses—a smaller, where death is caused by asphyxia or exhaustion, and a larger, where, even if its tendency thereto by asphyxia or exhaustion shall be averted, it will still certainly occur, by the special action of the poison on the histological structures it attacks.‖ Such a substance as Calabar bean may be employed with advantage to prevent death after the administration of the smaller quantity.

Physostigma has lately been proposed as a physiological antidote for atropia poisoning; and KLEINWÄCHTER has had the courage to employ it for this purpose, principally on the ground of its anti-mydriatic property.¶ As will be afterwards shown, these two substances appear to act in opposite modes on the ganglionic system of the blood-vessels; but the nature of their effects on the cerebro-spinal system is such as to make it irrational to anticipate any success in their employment as counter-agents.

* Rev. SAMUEL HAUGHTON, Dublin Quarterly Journal of Medical Science, August 1862.

† E. WOAKES, British Medical Journal, October 1860, &c.

‡ HARLEY, Lancet, 1856; L. VELLA, Comptes Rendus, 1860; CLAUDE BERNARD (opposes the view of counteraction), Leçons, &c., p. 377.

§ GRAVES, Clinical Lectures on the Practice of Medicine; ANDERSON, Effects of Belladona in Poisoning by Opium, 1854; LOPEZ, American Medico-Chirurgical Review, vol. iv. 1859; Dr W. F. NORRIS, American Journal of Medical Science, Oct. 1862; CAMUS (experimentally disproves this asserted antagonism), Gazette Hebdr., 11 Août 1865, and Canstatt's Jahresbericht, &c., Fünfter Band, 1866, p. 123.

‖ It has been found that frogs, after fatal doses of strychnia, may die *without any convulsions*, if care be taken to protect them from all causes of excitation—MARSHALL HALL, Aperçu du Système Spinal, p. 170; CLAUDE BERNARD, Lectures on Experimental Pathology and Operative Physiology, Medical Times and Gazette, 1860, v. ii., p. 25.

¶ Berliner klin. Wochschr, 38, 1864.

To argue from pupil effects alone of an antagonism between the actions of morphia and atropia, I believe to be absurd, while we know almost nothing of how iridal changes are produced by poisons. We shall never have antidotes to active substances until we can produce within the body chemical changes in their composition of such a nature as shall render them inert. By originating a secondary, and apparently counter, action, we may sometimes ward off death; but only where that would have been due to one of the symptoms of a small dose: we do not prevent the fatal action on the tissues of a large dose; and we run the risk of adding a second active substance, which cannot produce any effect without causing a tissue change, and which may, therefore, hasten and render more certain a previously doubtful, fatal result.

The most conspicuous symptom that is caused by physostigma is paralysis; and this necessarily depends on an effect produced on the nervous system, or on the muscular system, or on both. I believe this investigation is sufficient to show that it is due not to an action on the cerebral lobes, on afferent or efferent spinal nerves, or on muscles, but to one on the spinal cord, as a reflex centre. This spinal affection is the result of a primary and special action of Calabar bean; but it is more or less favoured by a simultaneous depression of the heart's action, as will be more conveniently illustrated in the special examination of the cardiac effects.

Action on the Heart.

The heart is affected in a marked manner by Calabar bean; and this has a more or less direct influence in causing death, according to the dose that may have been exhibited. With a large dose, the animal dies by cardiac syncope. With a smaller one, the heart beats are only diminished in frequency, and, as the circulation continues, the spinal cord is more and more affected, until its diastaltic function is destroyed and asphyxia caused. The latter effect is proved in the previous portion of this investigation, and it will be sufficient for the purpose of illustrating the former to quote one of my already published experiments with warm-blooded animals.

Experiment XLVI.

"The skin was raised in the left flank of a large black-and-white female cat, the needle-point of Wood's hypodermic syringe was inserted into the subcutaneous cellular tissue, and ten minims of a syrupy extract were injected" (equivalent to about four grains of the preparation usually employed in this investigation).

"In two minutes, trembling occurred; and, in three, the cat fell. Fluid escaped from the mouth, the pupils contracted, and urine was voided. In five minutes, the respirations became hurried, noisy and laboured. Reflex action could not be excited by severe stimulation, nor did the eyelids contract on irritation of the conjunctiva. The animal became perfectly flaccid, the only symptom of life was an occasional gasp, and this ceased entirely, seven minutes after the administration.

"*Autopsy*, immediate. The pupils were observed to dilate. A few contractions occurred in the muscles that were cut. *The heart was perfectly quiet, and without the slightest action.* . . . . On removing the pericardium, irregular movements occurred in the heart, and a partial contraction could be produced by irritation, fifteen minutes after death. The vessels of the

thorax and abdomen were well filled, and could be readily distinguished by the colour of their contents. On incising the left ventricle, blood of the usual arterial hue escaped; and on incising the right, dark blood appeared. Both were allowed to run side by side, when the contrast was distinctly shown."*

We have now to describe the various changes that are undergone by the heart before its contractions finally cease, and to examine the mechanism by which these changes are produced. For the former purpose, several experiments were performed on frogs whose hearts were exposed before the administration of the poison.

Experiment XLVII.—(*Temperature of Laboratory*, 58° *F.*)

A large frog, which weighed 730 grains, was fixed down on its back in such a manner that the circulation in the limbs was not to any extent impeded; and the heart was exposed by the removal of a portion of the sternum. This operation can easily be performed without injuring any large blood-vessel, and, indeed, without causing any loss of blood further than a temporary oozing from the cut surfaces. A few minutes afterwards, its heart-beats were frequently counted, and found to average seventy per minute. I then injected one grain of extract, mixed with a little water, into each thigh (two grains in all).

5 minutes afterwards,	cardiac contractions	= 64 per min.	
10 „	„	= 58 „	
15 „	„	= 43 „	Respirations ceased, except an occasional gasp.
20 „	„	= 39 „	
25 „	„	= 41 „	No respiratory movements.
30 „	„	= 37 „	
35 „	„	= 24 „	Contractions feeble.
40 „	„	= 22 „	
45 „	„	= 22 „	Skin much darker than originally.
50 „	„	= 19 „	
55 „	„	= 13 „	
1 hour 0 min.	„	= 14 „	
1 „ 5 „	„	= 12 „	Heart equally dark in systole and in diastole.
1 „ 10 „	„	= 10 „	
1 „ 15 „	„	= 9 „	
1 „ 20 „	„	= 9 „	
1 „ 30 „	„	= 10 „	
1 „ 40 „	„	= 10 „	Galvanism of sciatics caused neither direct nor reflex contractions. Muscles dark bluish, and active.
1 „ 50 „	„	= 8 „	Surface of heart opal blue in colour.
2 hours 0 „	„	= 8 „	Cardiac contractions extremely feeble.
2 „ 10 „	„	= 8 „	Skin deep olive-brown: the frog was originally a pale one.
2 „ 30 „	„	= 12 „	
3 „ 0 „	„	= 18 „	Spinal nerves still perfectly paralysed.
3 „ 30 „	„	= 21 „	
4 „ 0 „	„	= 24 „	Cardiac contractions quite synchronous, and diastole prolonged.
4 „ 30 „	„	= 26 „	A faint twitch occurred when the left sciatic was galvanised.

* *Op. cit.* section v. experiment iii.

The frog was now left in a cold and moist place until the following morning, when it was found jumping about actively, with its heart contracting forty-eight times per minute. It continued in very much the same condition for other two days, when it was killed.

This experiment is especially interesting because of the very near approach to death that was made. A decided effect was produced on the heart's action, as it was reduced in frequency by from seventy to eighty beats in the minute: respiratory movements were stopped; reflex spinal action was completely prevented; and the spinal motor nerves were, for many minutes, paralysed: and yet the animal revived; and regained all its lost functions except that of the heart, which only partially recovered itself. Such a result could never have been obtained with a warm-blooded animal, as death would soon have been produced by asphyxia, however long the heart might continue to contract. It is well known that the frog may live for many days after the removal of its lungs, as the respiratory function of those organs is shared in by the skin of this animal. The effects of the poison that was given in this case—and the dose was a very small one in proportion to the weight of the frog—had gradually disappeared, and the different tissues had returned to their former vitality, after having been acted upon for some time by a nearly normal blood-stream. Had the circulation ceased, or had the effects on the tissues been greater, and, therefore, more permanent, this return from pseudo-death could not have occurred.

Experiment XLVIII.—(*Temperature of Laboratory*, 58° *F.*)

The heart of a frog, weighing 396 grains, was exposed. After allowing a few minutes for recovery from shock, the number of the cardiac contractions was determined at intervals of five minutes, during twenty minutes, and found to vary little from forty-two beats per minute. One grain of extract, in a few drops of water, was then injected into the subcutaneous tissue of each thigh (two grains in all).

2 minutes afterwards,	cardiac contractions	= 36	per min.	
4 „	„	= 24	„	
6 „	„	= 20	„	Contractions seem feeble, and heart is not so pale during systole as is normal.
8 „	„	= 17	„	
10 „	„	= 17	„	
12 „	„	= 15	„	Heart walls seem nearly as dark during systole as during diastole.
14 „	„	= 14	„	
16 „	„	= 14	„	
18 „	„	= 15	„	Frog is helpless and flaccid.
20 „	„	= 14	„	
25 „	„	= 14	„	Almost no difference of colour during systole, and a great prolongation of diastole.
30 „	„	= 12	„	
35 „	„	= 13	„	
40 „	„	= 12	„	
45 „	„	= 8	„	The ventricular contractions seem more feeble than the auricular.

50 minutes afterwards, cardiac contractions	= 8 per min.		
1 hour	,,	= 8 ,,	The ventricular contraction is scarcely perceptible.
1 ,, 5 min.	,,	are irregular; there being three ventricular contractions per minute, and six auricular.	

This irregularity continued for other ten minutes, and then became greater, only one contraction per minute, of the most feeble character, occurring in the ventricles for two and sometimes three in the auricles. During the period of inaction, the heart rested in diastole.

1 hour 20 minutes after poisoning, cardiac contractions			= 0 per min.	
1 ,, 40 ,,	,,	,,	= 8 ,,	All the chambers contract, with long intervals, and in a vermicular manner.
1 ,, 50 ,,	,,	,,	= 12 ,,	
2 hours	,,	,,	= 12 ,,	
2 ,, 10 ,,	,,	,,	= 12 ,,	
3 ,, 15 ,,	,,	,,	Again irregular, and exactly as at one hour and five minutes after the poisoning.	
3 ,, 40 ,,	,,	,,	The irregularity was still further increased, the ventricles being occasionally quite motionless, in a dilated condition, for one minute, while both auricles contracted six and sometimes eight times. The contractions consisted merely of feeble wave-like movements of the different chambers. The surface of the heart, and, still more, the pericardium had for some time assumed a blue colour.	

When the heart was again examined, twenty-two hours after poisoning, there was no spontaneous movement. It was then gently stimulated by an interrupted galvanic current, and a very feeble contraction of all the chambers followed, which did not repeat itself, but which could be reproduced by a renewal of the stimulation. For some time after these irritations, feeble and irregular contractions occasionally occurred, one ventricle contracting, and, after several minutes, two or three auricular movements following, but with considerable pauses. Forty-eight hours after the administration, the heart was motionless, dark and dilated; gentle galvanism produced no effect, except that the ventricles became rather paler, but a moderate current caused a contraction of all its chambers, succeeded by perfect quiet.

In sixty hours, the heart was pale and quiet, and no movement could be produced by galvanism. Notwithstanding the paleness of its walls, the heart was not contracted, as its chambers contained a considerable quantity of dark, fluid blood. The colour of its walls was, therefore, due to contraction of their capillaries, caused either by the rigor of death, or, as I am more inclined to believe, by the galvanic stimulation they had been so frequently subjected to. At this time, all the muscles were stiff.

Experiment XLIX.—(*Temperature of Laboratory*, 57° *F.*)

The average number of cardiac contractions in a frog, weighing 415 grains, was fifty-five per minute. Two grains and a-half of extract, in fifteen minims of distilled water, were injected into the subcutaneous tissue of each thigh (five grains altogether).

2 minutes afterwards, cardiac contractions		= 52 per min.	
4 ,,	,,	= 50 ,,	
6 ,,	,,	= 24 ,,	
8 ,,	,,	= 19 ,,	
10 ,,	,,	= 15 ,,	
12 ,,	,,	= 14 ,,	No chest respiratory movements.

14 minutes afterwards,	cardiac contractions	= 14	per min.	Systole not quite so pale as originally.
16 „	„	= 13	„	Frog is quite flaccid; it cannot turn from its back, but occasional faint struggles occur.
18 „	„	= 11	„	Contractions very feeble.
20 „	„	= 10	„	Heart as dark during contraction of the ventricles as during dilatation of them.
22 „	„	= 8	„	Irregular and non-synchronous; the auricles sometimes contracting nine or eleven times per minute.
24 „	„	= 8	„	of ventricles, 10 of auricles.
26 „	„	= 8	„	„ 14 „
28 „	„	= 7	„	„ 14 „
30 „	„	= 8	„	„ 14 „
32 „	„	= 0		during one minute, and then a few feeble contractions in the auricles, and again a pause.
34 „	„	= 0	per min.	
36 „	„	= 0	„	
38 „	„	= 0	„	of ventricles, 3 of auricles.
40 „	„	= 8	„	„ 12 „
42 „	„	= 8	„	„ 16 „
45 „	„	= 8	„	„ 16 „
50 „	„	= 9	„	Synchronous.
55 „	„	= 9	„	Skin extremely dark.
1 hour 0 min.	„	= 8	„	No reflex movement on stimulation. Galvanism of the sciatic nerves causes faint twitches of the toes.
1 „ 10 „	„	= 8	„	
1 „ 15 „	„	= 8	„	
1 „ 20 „	„	= 8	„	All the spinal nerves are paralysed; the muscles remaining active and blue.
2 hours 0 „	„	= 8	„	
3 „ 0 „	„	= 8	„	

The observations were now interrupted until twenty hours after the exhibition of the poison, and, at this time, the frog was quite flaccid and dark; the heart was contracting sixteen times in the minute, and the striped muscles were irritable, but otherwise the animal was quite dead. Twenty-eight hours after the exhibition of Calabar bean, the cardiac contractions were seventeen per minute; and at thirty hours they were ten. In forty hours, the heart was found perfectly still, dark and somewhat dilated. Direct galvanism produced a slow contraction of the portion of cardiac muscle included in the circuit; it did not excite a normal heart beat. The striped muscles contracted very sluggishly when galvanised. This idio-muscular contractility was retained in the heart until sixty-nine hours, and in the striped muscles till seventy hours after the administration of the poison. They then became pale, stiff and acid.

In this experiment, a large proportion of the extract had not been absorbed, but escaped from the thighs when the skin was incised to allow the sciatic nerves to be exposed. The effects cannot, therefore, be regarded as those produced by five grains, but must be held to have been caused by a much smaller quantity. Indeed, where three grains were absorbed by a frog of nearly the same weight, the action on the heart was much more decided and marked.

Experiment L.—(*Temperature of Laboratory*, 60° *F.*)

The exposed heart of a frog, weighing 469 grains, was found to contract, on an average, sixty-seven times in the minute. One grain of extract, in five minims of water, was then injected into each thigh, and a third grain was injected into the stomach by means of a narrow caoutchouc tube (total three grains).

2 minutes afterwards,	cardiac contractions	= 60	per min.	
4 „	„	= 54	„	
6 „	„	= 34	„	
8 „	„	= 30	„	
10 „	„	= 21	„	Very feeble. Heart not quite so pale, now, in systole.
12 „	„	= 13	„	
14 „	„	= 11	„	Frog jumped about when set free, but soon fell on its back, and remained there.
16 „	„	= 8	„	
18 „	„	= 0	„	Colour of frog has changed from yellowish brown to dark bronze.
20 „	„	= 0	„	
22 „	„	= 0	„	of the ventricles, 2 of the auricles.
24 „	„	= 0	„	
28 „	„	= 0	„	
30 „	„	= 0	„	

After this, an occasional auricular contraction occurred, but no spontaneous movement was seen of the ventricles. The heart rested in diastole, and was of a dark colour with a bluish tinge. For twenty minutes after this, a slight stimulus excited a few rhythmical contractions, followed, in a few seconds, by rest in diastole, but capable of being reproduced in the same way. One hour and ten minutes after the administration, no reflex movement could be excited; but the sciatic nerves continued active for longer than thirty hours—the rapid stoppage of the circulation having prevented that prolonged contact with the poison that is necessary for the paralysis of the endorgans. The striped muscles remained contractile until more than forty-eight hours, and contractility was also retained by the cardiac muscle during this period.

A possible source of fallacy is introduced into these experiments by the operations that they require. It seemed, therefore, of some importance to test the effects on the frog's heart of mere exposure to the air. For this purpose, a portion of the sternum was removed from an active frog, in exactly the same manner as in the last four experiments, with the following result:—

5 minutes afterwards,	cardiac contractions	= 62.
10 „	„	= 61.
20 „	„	= 62.
30 „	„	= 59.
40 „	„	= 60.
50 „	„	= 59.
1 hour 0 minutes,	„	= 55.
1 „ 30 „	„	= 57.
2 „ 0 „	„	= 60.
2 „ 30 „	„	= 56.
3 „ 0 „	„	= 58.
3 „ 30 „	„	= 59.
4 „ 0 „	„	= 59.

It is almost needless to remark that the change in the systolic colour of the heart, from pale to dark, did not occur. The heart's surface was prevented from drying by an occasional drop of water; and this was also done during the experiments in which physostigma was administered.

We have now sufficiently described, as proposed (p. 743), "the various changes that are undergone by the heart before its contractions finally cease;" and from the data given, these may be summarised in their order of occurrence as follows:—

1st, Diminution never preceded by increase, of the frequency of the contractions, with prolongation of the period of rest; 2d, Feebleness of the contractions, with no change of colour on the occurrence of systole; 3d, Irregularity of rhythm, the auricles contracting more frequently than the ventricles, and, for intervals, contracting alone; 4th, Stoppage of all the heart's chambers—If the poison be absorbed quickly and in large quantity, the fifth and sixth effects may not occur; 5th, Renewal of contractions, either by all the chambers at once, or by one or more in the first place; 6th, Gradual recovery to a low rate of action, and continuance at this for from a few minutes to several days; 7th, Stoppage *in diastole* of spontaneous contractions; and 8th, Loss of the idio-muscular irritability of the heart, rigor and change of reaction from alkaline to acid.

This method of affecting the heart distinguishes physostigma from the great majority of cardiac poisons, which may be well represented by *Antiaris toxicaria,** *Tanghinia venenifera,*† *Digitalis,*‡ *Helleborus niger*, *H. viridis*§ and the green resin obtained from *Nerium Oleander*.‖ These produce first irregularity and acceleration of the heart's action, then a diminished frequency, *caused by protraction of the ventricular systole,* and, finally, stoppage of the contractions by "*cessation of the dilatation of the ventricles,* which then remain contracted, white and perfectly empty."¶ In producing cardiac paralysis, physostigma acts in a manner exactly the reverse. It causes no acceleration, it diminishes the frequency of the contractions *by prolonging the ventricular diastole,* and it produces the final stoppage *by cessation of the contraction of the ventricles,* which then remain dilated, dark and full of blood. Very small doses of digitaline and the alcoholico-aqueous extract of *Nerium Oleander* are said to act on the heart in a manner which seems to resemble closely that of Calabar bean,** but no other cardiac poisons appear to share in its peculiarities.

It now remains that we examine the mechanism by which these changes are produced, and endeavour to determine what tissues or structures are influenced by physostigma to effect them. For this purpose, it will be necessary to investi-

* KÖLLIKER, VULPIAN, CLAUDE BERNARD and others, in various papers.

† M. EUG. PÉLIKAN et Dr DYBKOWSKI; Recherches physiologo-toxicologiques sur l'action de quelques poisons du cœur; and Comptes Rendus, 1865, p. 1209.

‡ *Ibid.* § *Ibid.* ‖ *Ibid.*

¶ On the Application of Physiological Tests for Certain Organic Poisons, and especially Digitaline, by C. HILTON FAGGE, M.D., and THOMAS STEVENSON, M.D. GUY'S Hospital Reports, 3d series, 1866, vol. xii. p. 47.

** Nouvelles Recherches sur le poison du *Nerium Oleander*. Note de M. EUG. PÉLIKAN. Comptes Rendus, 1866, p. 237.

gate the possible influences of the cerebro-spinal nervous system, whether exerted through the vagi or through the spinal nerves, and the possible influences of the sympathetic system, whether exerted through the great sympathetic trunks and their branches or through the ganglia contained in the heart's substance. Any effect on idio-muscular contractility has been already abundantly disproved; but it will be necessary to observe how far the impairment and cessation of respiration may explain the cardiac effects in warm-blooded animals.

The paralysis of the heart in diastole and the diminution in the frequency of its contractions by protracted periods of rest in a *dilated condition*, as well as the frequent renewal of its action after a long pause in diastole, might, in the first place, suggest that the inhibitory function of the vagi nerves was being exerted. On this account, it may be advisable to examine their condition during Calabar bean poisoning.

Experiment LI.

An active frog was selected, of the weight of 863 grains, and its heart and two vagi nerves were exposed. The latter were separately tested by galvanism, and each produced stoppage of the heart's action, in diastole. A few minutes afterwards, the average of the heart's contractions was ascertained to be fifty-eight in the minute. Three grains of extract, in a few drops of water, were injected into the subcutaneous tissue of the two thighs—one-half into each.

2 minutes afterwards,	cardiac contractions = 54 per min.	
4 „	„ = 47 „	
6 „	„ = 40 „	
8 „	„ = 37 „	
10 „	„ = 36 „	
12 „	galvanism of the left vagus produced stoppage in diastole.	
14 „	cardiac contractions = 34 per min.	
16 „	„ = 33 „	Contractions feeble.
18 „	„ = 30 „	Heart dark in systole, as well as in diastole.
20 „	„ = 24 „	
25 „	„ = 20 „	
30 „	„ = 19 „	
35 „	galvanism of the left vagus produced stoppage for one minute.	
40 „	cardiac contractions = 15 per min.	
45 „	„ = 18 „	
50 „	„ = 14 „	Marked pause between auricular and ventricular contractions.
1 hour 0 min.	„ = 12 „	
1 „ 5 „	galvanism of the left vagus produced stoppage of all the chambers.	
1 „ 10 „	cardiac contractions were irregular, on being renewed; only twelve ventricular now occurring to eighteen auricular, per minute.	
1 „ 20 „	cardiac contractions continue, with the above irregularity.	
1 „ 30 „	cardiac contractions still irregular; galvanism of the left vagus produced stoppage of all the chambers, but a stronger current was required than before.	
1 „ 32 „	galvanism of either sciatic nerve caused movements confined to the limb whose nerve was stimulated.	
1 „ 40 „	cardiac contractions continue irregular, as above.	

1 hour 50 min. afterwards,	galvanism of the left vagus produced stoppage of all the chambers; and the heart remained at rest, in diastole, for ten minutes, when a feeble auricular contraction occurred, and, soon after, a ventricular.
1 ,, 55 ,,	galvanism of either sciatic nerve caused merely faint twitches of the toes.
2 ,, 10 ,,	cardiac contractions = 18 per minute; very feeble, but rhythmical.
2 ,, 15 ,,	strong and continued galvanism of the left vagus produced no effect on the cardiac contractions.
2 ,, 18 ,,	strong and continued galvanism of the right vagus, which had not been irritated during the experiment, produced no effect on the cardiac contractions.
2 ,, 20 ,,	strong galvanism applied to either sciatic nerve produced no effect.
2 ,, 40 ,,	cardiac contractions stopped. Irritation could produce a few contractions for only a few minutes longer.

From this experiment it is apparent that the vagi retain their inhibitory power over the heart during the whole period that its action is being modified by physostigma. Ultimately, however, they are themselves paralysed, as might be anticipated from the analogies that exist between them and the spinal nerves; and, as this and many other similar experiments prove, the functions of the vagi and of the spinal motor nerves are lost simultaneously, or nearly so.

To illustrate this in warm-blooded animals, it is necessary to exhibit so small a dose of the poison that death shall be caused by asphyxia, and the heart afterwards continue to contract, however irregularly.

Experiment LII.

Half a grain of extract, suspended in ten minims of distilled water, was injected under the skin on the back of a full-grown and active rabbit. The usual symptoms followed; and, in thirty-four minutes, the animal was dead, all respiratory movement having ceased. Immediately afterwards, the heart was exposed; and it was found contracting in normal rhythm, twenty-two times in the minute. The right vagus was divided; and the end proximal to the heart was galvanised, with the effect of producing an immediate stoppage, in diastole, for several seconds. The vagi were tested, occasionally, during twenty minutes after the rabbit's death, and their cardiac inhibitory function, as well as their excito-motory power over the stomach and œsophagus, continued active all this time; and, for the same period, the sciatic, intercostal, phrenic and other spinal nerves retained their motor conductivity. Twenty-two minutes after death, the heart was contracting eight times per minute. Galvanism of either vagus could now neither stop the cardiac action nor excite œsophageal or gastric movements. The sciatic and other spinal nerves were tested at twenty-five minutes after death, and found to be perfectly paralysed.

It is, therefore, quite possible, as far, at least, as conveyance by the vagi nerves is concerned, for Calabar bean to act on the heart by exciting the cardiac inhibitory centre in the medulla oblongata. But, if this be the method of its action, the prevention of this possible influence, by division or previous paralysis of the vagi, or by destruction of the medulla oblongata, should render it impossible for Calabar bean to produce its usual effects on the heart.

Experiment LIII.

The heart and the two vagi nerves were exposed and the latter divided in a frog, weighing 700 grains, and, a few minutes later, the cardiac contractions were found to have an average of sixty-six in the minute. Two grains of extract, in ten minims of distilled water, were then injected into each thigh (altogether four grains).

2 minutes afterwards,	cardiac contractions	= 60 per min.		
4 ,,	,,	= 60	,,	
6 ,,	,,	= 53	,,	
8 ,,	,,	= 46	,,	
10 ,,	,,	= 39	,,	
12 ,,	,,	= 31	,,	
14 ,,	,,	= 27	,,	
16 ,,	,,	= 24	,,	
18 ,,	,,	= 22	,,	
20 ,,	,,	= 18	,,	
22 ,,	,,	= 16	,,	All the chambers are of dark colour, even in systole.
24 ,,	,,	= 13	,,	
26 ,,	,,	= 14	,,	
28 ,,	,,	= 12	,,	Ventricular contraction is extremely feeble.
30 ,,	,,	= 10	,,	
35 ,,	,,	= 9	,,	
40 ,,	,,	irregular; six ventricular for twelve auricular.		
42 ,,	,,	stopped in diastole, for thirty-five seconds.		

The heart then resumed its previous unrhythmical action of six ventricular to twelve auricular movements in the minute, and continued to contract, with various changes of irregularity, for many hours longer.

It would needlessly occupy space, were I to narrate any other of the many experiments that were performed with frogs whose vagi had been divided; as the results, and even the details, were in all of them very much the same. It might be proper to instance, at this place, an experiment of the same character on a warm-blooded animal, but I content myself by referring to Experiment LXI., at page 763.

In curare we possess an agent that, within a few minutes after its exhibition, produces complete paralysis of all the motor nerves, including the vagi.* It will, therefore, act as effectually as division, in preventing any inhibitory influence being exerted on the heart.

Experiment LIV.

A small dose of curare was inserted under the skin of a frog's back. Ten minutes afterwards, the animal was perfectly paralysed. At twenty minutes, its heart was exposed, and observed to be contracting fifty-four times in the minute. The vagi and sciatic nerves were tested by galvanism, and their conductivity was found to be completely destroyed. Three grains of extract of physostigma, in water, were injected into the two thighs. The usual cardiac effects were produced within the time that might have been expected, from such a dose; irregularity of the rhythm having occurred in forty-four minutes, and final paralysis, with all the chambers dark, full and dilated, in a few minutes later.

* CLAUDE BERNARD, Leçons sur les effects des Subs. Tox. &c., p. 352; KÖLLIKER, *loc. cit.*; MARTIN-MAGRON, Journal de la Physiologie, 1859, p. 649, &c.

These are sufficient to prove that Calabar bean influences the heart neither by an action on the inhibitory centres nor by one on the nerves that connect those centres with it. I think they are also sufficient to exclude the rest of the cerebro-spinal nervous system: as the only other spinal nerves connected with the heart, by the branches of the great sympathetic trunks or otherwise, are either sensory, or at any rate afferent, nerves that form through the cord reflex arrangements with the vagi and vaso-motor nerves; or excito-motory ones whose action is to increase the frequency of the heart's contractions, but whose function is only periodically exerted, and is quite unnecessary for the continuance of the ordinary rhythmical occurrence of those contractions.* But, to remove the possibility of any doubt on this point, the following experiment was performed:—

Experiment LV.

The spinal cord was divided between the occiput and first vertebra of a frog, weighing 376 grains, and a wire was passed down the spinal canal, so as to produce complete paralysis, and into the cranial cavity, so as thoroughly to break up and destroy the brain. A short time thereafter, the heart was contracting at the rate of forty beats per minute. One half of three grains of extract, in twenty minims of water, was injected into each thigh.

2	minutes afterwards,	cardiac contractions	= 37	per min.
4	„	„	= 31	„
6	„	„	= 28	„
8	„	„	= 26	„
10	„	„	= 25	„
12	„	„	= 24	„
14	„	„	= 22	„
16	„	„	= 20	„
18	„	„	= 17	„
20	„	„	= 14	„
22	„	„	= 13	„
24	„	„	= 12	„

* Dr POWER observes, in the sixth edition of CARPENTER'S "Principles of Human Physiology" (page 217, note), that "the essential cause of the rhythmical action of the heart must still remain an unsolved question." The exact influence of the various nerves that connect the heart with the central nervous systems, appears to be quite as imperfectly ascertained, judging by the contradictory statements and deductions of eminent physiologists. LEGALLOIS and PHILIP WILSON, and, afterwards, BUDGE, SCHIFF, REID, WEBER, MOLESCHOTT, VON BEZOLD and others, have shown that a connection certainly exists; but they have left the details of the question unsettled by the great differences in many of their opinions, as, for example, on the cardiac functions of the vagi. VON BEZOLD, in 1863, attempted to prove the existence, in the spinal cord, of an excito-motory centre, whose stimulation increases not only the number of the beats, but also the blood-pressure—the latter being due to augmented force in the heart's contractions. LUDWIG and THIRY opposed this opinion, and asserted that the increased blood-tension is really an effect of excitation of the vaso-motor nerves. In a recent investigation (Comptes Rendus, 25 Mars 1867), MM. E. and M. CYON give their adherence to the views of LUDWIG and THIRY. They also attempt to show that the spinal cord, through the sympathetic system, supplies the heart with nerves that possess the power of directly accelerating its contractions, and that are antagonistic to the vagi, in that, while the latter diminish the frequency and increase the force of the contractions, the spinal "nerfs accélérateurs," on the other hand, increase the frequency and diminish the force.—*April* 1867.

26 minutes afterwards, cardiac contractions	= 9 per min.		
28 „ „	= 9 „	Very feeble. Heart dark during systole.	
30 „ „	= 7 „		
35 „ „	irregular; four ventricular for six auricular.		

Irregular and unrhythmical contractions continued for many hours afterwards. The final stoppage was in diastole.

It may now be useful to examine what connection exists, as cause and effect, between the impairment and cessation of the respiratory movements and the interference with, and stoppage of, the cardiac contractions, especially as so deservedly distinguished a physiologist as HARLEY has asserted that Calabar bean is a respiratory poison purely, which causes death by destroying the conductivity of the motor nerves of respiration. CHRISTISON, in his investigation, was the first to observe an action on the heart, and he believes that death is caused by paralysis of that organ. My former results were, so far, in accordance with this statement; but they also show that death may often be due to asphyxia, and while I agree with HARLEY in this, I believe him to be in error when he asserts that paralysis of the motor nerves is the cause of such death. The data that have already been given are sufficient to prove that motor nerve conductivity is always retained, in warm-blooded animals as well as in cold, for many minutes after the complete stoppage of respiratory movements, and that such stoppage is due to destruction of the reflex and co-ordinating functions of the medullæ spinalis and oblongata. To complete this evidence, it will be sufficient to show that no connection of cause and effect necessarily exists between the impaired respiratory movements and the cardiac paralysis.

Experiment LVI.

In a large retriever dog, it was found that the mean number of respirations was ten, and the mean number of cardiac contractions 126, during seven minutes immediately preceding the injection of six grains of extract, suspended in water, into the right jugular vein.

1 min.	after the injection, the respirations	= 10, cardiac contractions	= 78 per min.
1 „ 30 sec.	„ „	= 11, „	= 54 „
2 „ 2 „	„ „	= 9, „	= 40 „
3 „ 0 „	„ „	= 9, „	= 8 „
7 „ 0 „	„ „	= 10, „	= 20 „
9 „ 0 „	„ „	= 10, „	= 16 „
10 „ 0 „	„ „	= 9, „	= 9 „
10 „ 30 „	„ „	= 0, „	= 0 „

This experiment gives the result that, in one minute and thirty seconds after the poison was administered, the number of cardiac contractions had fallen to less than one-half, while the respiratory movements had increased by one per minute; and it distinctly shows the absence of any respiratory change to cause the marked effects that were produced on the heart's action.

Experiment LVII.

A frog, weighing 460 grains, had its heart exposed by removing a small portion of the sternum. It was acting at the rate of forty-eight beats per minute, while the respirations were seventy-two. Five minutes afterwards, the heart was contracting at the rate of forty-five per minute, while the respirations were seventy-four. One grain and a half of extract, suspended in water, was injected under the skin of each thigh (three grains in all).

In 5 min.,	heart = 36 per min.		Respirations = 74 per min.
10 ,,	,, = 28 ,,		,, = 64 ,,
15 ,,	,, = 22 ,,		,, = 63 ,,
20 ,,	,, = 14 ,,	Contractions weak; and heart continues dark during systole.	,, = Frequent gasping movements, which cannot be counted.
25 ,,	,, = 13 ,,		,, =
30 ,,	,, = 12 ,,		An occasional gasp.
40 ,,	,, = 10 ,,		,, have stopped.
50 ,,	,, = 10 ,,		
1 h. 0	,, = 8 ,,	Contractions very feeble.	
1 ,, 10	,, = 8 ,,		
1 ,, 15	,, = 0 ,,	Stopped in diastole.	
1 ,, 20	,, = 0 ,,		
1 ,, 30	,, = 12 ,,		
1 ,, 40	,, = 12 ,,		
1 ,, 50	,, = 12 ,,		
2 ,, 0	,, = 10 ,,		
2 ,, 30	,, = 8 ,,	Contractions irregular; two auricular for one ventricular.	

These two experiments prove distinctly that Calabar bean has a direct influence on the heart, that is quite independent of the indirect influence it exerts on that organ by arresting the respiratory movements. Such arrest does, doubtless, assist the action on the heart, especially during the later stages of the poisoning, by impeding the circulation; and in mammals, when a small dose has been exhibited, such comparatively slight diminution as is first produced in the frequency of the heart's contractions must even be partly caused by the early retardation and cessation of the respiratory movements that constantly occur.

The cardiac action of physostigma is, thus, quite independent of the cerebro-spinal nervous system, and is not a mere effect of the paralysis of respiration. It must, therefore, be caused by an action of a direct nature on the cardiac ganglia, which seem to be the only constant exciters of this organ, however its contractions may be *regulated* by other nerves. The peculiar changes that the heart's action undergoes—the diminution in the frequency of its beats, then their stoppage or irregularity, sometimes followed by renewal of rhythmical contractions, or of independent movements in all the chambers, or in one only—prove that Calabar bean first diminishes the vitality of the exciting ganglia, and then paralyses them. Their influence is, at any rate, maintained until spontaneous movements cease; for, if we divide the ventricles from the auricles at a late stage of the poisoning and when the contractions are unrhythmical, those chambers alone

that are still ir connection with the great exciter ganglia in the auriculo-ventricular septum will continue their spontaneous movements.

The action is not a very powerful one, and its characteristics may be explained because of that; for the effects of physostigma on the heart appear to be similar to those of *weak forms* of at least two of the ordinary cardiac poisons—of digitaline in minute doses, and of the alcoholico-aqueous extract of *Nerium Oleander*, which is merely the green extract mixed with various impurities, without which it has the ordinary actions of the larger class of those substances that affect the heart.*

ACTION ON THE BLOOD-VESSELS.

The question of the action of Calabar bean on the condition of the vascular system is intimately connected with that of the cardiac effects; and must, therefore, be considered also, that we may complete the examination of the influence of this substance on the circulation. For this purpose, I instituted two sets of experiments. In the first (1), the blood-tension in the arterial and venous systems was observed; and in the second (2), the calibre-changes of the smaller blood-vessels and of the capillaries were investigated.

1. *Examination of the Blood-Tension, and of the Coincident Changes in the Cardiac and Respiratory Movements, and in the Temperature.*

In these experiments, the tension in the arterial system was determined by dividing one of the carotid arteries and connecting the end proximal to the heart with a modification of POISSEUILLE's hæmadynamometer, in which two indicating columns were connected with the reservoir. The tube of one of these had an extremely small orifice where it dipped into the mercury, and it, therefore, registered the mean pressure only. The orifice of the other had the same diameter as the rest of the tube, and the contained mercury had, therefore, immediately communicated to it every change of pressure, and oscillated synchronously with the heart's beats. The venous pressure was ascertained by a simple hæmadynamometer having one registering column. The indicating columns were divided into inches and tenths of inches.

In the experiments where the temperature was observed, a delicate thermometer was inserted into the subcutaneous tissue at the flank of the animal, and retained there, under the charge of an assistant, during all the time of the experiment.

* PÉLIKAN, *op. cit.*, Comptes Rendus, 1866.

a. *Arterial Tension only.*

Experiment LVIII.

A large and vigorous retriever dog was placed on a table, and tied down by its four limbs.*

Time.	Time after Administration.	Mean Pressure.	Pressure Oscillates between	No. of Cardiac Contractions.	No. of Respirations.	Notes of Operations and of Symptoms.
H. M. S.	M. S.			Per Min.	Per Min.	
3 16 0 P.M.	...	...	...	...	20	The right carotid artery was exposed and connected with the hæmadynamometer.
3 16 30	...	6·8	5·2 & 9·0	...	...	
3 19 0	...	7·0	5·0 9·0	86	18	
3 20 30	...	6·9	5·3 9·3	...	18	
3 22 0	...	7·2	5·3 9·3	108	19	
3 24 0	...	7·0	5·1 8·9	...	...	
3 26 20	...	...	...	...	...	Two grains of extract, in forty minims of distilled water, were injected into the subcutaneous tissue of the abdomen.
3 26 30	...	6·8	5·5 9·0	102	...	
3 26 50	...	...	...	...	...	The injection of the poison was completed.
3 29 0	2 10	6·9	4·9 9·2	93	19	Dog is perfectly quiet.
3 33 0	6 10	6·8	4·5 9·2	90	18	A few struggles occurred.
3 35 0	8 10	6·7	4·9 8·5	75	17	
3 38 30	11 40	7·5	5·0 10·0	90	19	Somewhat violent struggles.
3 41 0	14 10	7·8	5·5 9·3	90	15	Muscular twitches over nearly all the body.
3 41 30	14 40	8·0	...	...	17	
3 42 0	15 10	8·4	7·0 10·0	...	21	Muscular twitches have increased in violence.
3 43 0	16 10	8·9	7·0 10·0	...	22	
3 44 0	17 10	8·8	7·5 10·4	...	...	
3 45 0	18 10	8·5	7·0 9·3	106	...	
3 46 0	19 10	8·3	7·0 9·3	104	31	
3 48 0	21 10	8·2	6·5 9·5	98	38	Twitches are now much feebler; great lachrymation and salivation.
3 50 0	23 10	8·15	7·5 9·5	...	9	
3 52 0	25 10	7·9	6·5 9·4	96	40	
3 55 0	28 10	7·65	6·5 9·9	96	37	
3 57 0	30 10	7·55	6·5 8·6	...	30	Urine and fæces passed.
3 58 0	31 10	...	...	...	34	Respirations noisy, from mucus in the trachea.
3 59 0	32 10	7·3	6·0 8·5	100	28	Very fluid fæces passed.
4 0 0	33 10	7·2	6·3 8·5	112	25	
4 3 30	36 40	7·1	6·0 8·0	112	22	The twitches have very much diminished in force.

* I have much pleasure in acknowledging the valuable assistance I derived in this series of experiments from my friend, Dr GAMGEE. I am also indebted, for essential aid in all or several of them, to Dr BRUNTON, and to Messrs PATON, RITCHIE, FINLAY, CATON, HOGG, HOLDEN, WRIGHT, HARDIE, GAIRDNER, GLASCOTT, LUCAS, M'EWAN, HOWIESON, CROMBIE and YOUNG. Without the co-operation of these gentlemen, it would have been quite impossible to obtain the many simultaneous observations which are contained in these experiments, and to which much of their value is due.

Experiment LVIII.—continued.

Time.	Time after Administration.	Mean Pressure.	Pressure Oscillates between	No. of Cardiac Contractions.	No. of Respirations.	Notes of Operations and of Symptoms.
H. M. S.	M. S.			Per Min.	Per Min.	
4 6 0 P.M.	39 10	7·0	...	...	24	
4 7 0	40 10	6·9	6·0 & 7·8	116	23	
4 10 0	43 10	6·8	5·8 7·5	120	22	The twitches are now few, faint and occasional.
4 16 0	49 10	6·8	5·6 7·3	128	23	
4 17 30	50 40	6·7	5·8 7·3	...	22	The dog appears almost perfectly well.
4 19 0	...	6·9	5·5 7·5	124	...	It was deemed advisable to administer a second dose at this stage.
4 22 0	...	...	...	...	...	Commenced the injection of four grains of extract, in thirty minims of distilled water, into the subcutaneous tissue of the right flank.
	Time after Second Administration.					
4 22 30	...	...	...	122	21	The injection was completed.
4 23 0	0 30	6·8	5·8 7·5	...	22	
4 24 0	1 30	6·7	6·0 7·5	...	23	
4 27 0	4 30	7·0	5·8 7·5	120	22	Strong muscular twitches have reappeared.
4 28 0	5 30	6·75	6·0 8·0	116	21	Liquid fæces passed, and urine "jetted" out in a full and abundant stream
4 30 0	7 30	7·0	...	...	21	
4 30 30	8 0	7·35	...	...	...	
4 31 0	8 30	6·9	6·3 8·3	120	...	Respirations are rather laboured.
4 33 0	10 30	6·9	6·0 8·0	...	22	
4 34 0	11 30	7·1	6·0 8·0	118	...	
4 35 0	12 30	7·2	6·3 8·0	...	...	
4 37 0	14 30	7·4	6·0 8·0	...	23	
4 38 0	15 30	7·3	6·5 8·3	116	22	
4 42 0	19 30	7·6	6·0 8·3	120	24	Twitches are so strong as to cause frequent slight spasms.
4 44 0	21 30	7·15	6·4 7·8	112	25	
4 46 30	24 0	7·2	6·5 8·5	...	...	Urine and fæces discharged copiously.
4 47 0	24 30	7·5	6·5 9·5	...	23	
4 48 0	25 30	7·8	6·0 11·0	92	26	
4 49 30	27 0	7·3	5·6 10·0	30	...	
4 50 0	27 30	7·1	...	...	...	
4 50 15	27 45	6·9	...	...	...	
4 50 30	28 0	6·7	5·0 8·5	...	23	
4 50 31	28 1	6·2	...	...	...	
4 50 35	28 5	5·9	...	...	...	
4 50 38	28 8	5·8	...	...	...	
4 50 40	28 10	5·4	4·0 7·0	40	...	
4 51 0	28 30	5·4	4·0 4·7	...	...	The twitches continue with considerable strength.
4 51 30	29 0	6·9	...	...	25	

Experiment LVIII.—continued.

Time.	Time after Second Administration.	Mean Pressure.	Pressure Oscillates between		No. of Cardiac Contractions.	No. of Respirations.	Notes of Operations and of Symptoms.
H. M. S.	M. S.				Per Min.	Per Min.	
4 52 0	29 30	7·2	6·0 &	9·0	...	...	
4 53 0	30 30	7·1	5·9	9·0	...	...	
4 54 0	31 30	6·3	5·9	8·2	...	...	
4 55 0	32 30	6·3	5·0	8·0	24	...	
4 55 20	32 50	5·6	4·0	8·0	...	...	
4 55 30	33 0	6·1	5·0	8·0	...	...	
4 56 0	33 30	5·0	4·5	7·0	...	21	
4 57 0	34 30	4·6	3·5	5·8	22	...	
4 57 15	34 45	4·0	3·5	5·5	...	...	
4 57 30	35 0	4·4	3·5	6·0	...	...	
4 58 0	35 30	4·4	3·0	6·0	...	27	The respirations are extremely shallow and gasping.
4 59 0	36 30	4·0	3·0	5·0	...	...	
4 59 30	37 0	3·5	2·5	4·5	...	...	
5 0 0	37 30	2·5	1·0	3·0	...	...	
5 0 30	38 0	1·8	1·0	2·5	18	4	Respirations are mere gasps.
5 1 15	38 45	1·6	1·0	2·0	...	4	
5 1 30	39 0		1·0	1·5	...	...	Mercurial column has fallen into the reservoir.

In the autopsy, which was immediately made, the heart was found dilated with dark blood. Occasional contractions occurred for twenty minutes after death. The sciatic, intercostal and phrenic nerves were active, and galvanism of a vagus nerve excited vermicular movements of the stomach. The intestinal peristalsis was feeble. Galvanism of the cervical sympathetics produced no contraction of the pupils. A quivering movement continued in the striped muscles for many minutes.

The first portion of this experiment shows that after the administration of the poison a very distinct rise occurs in the arterial tension, while the number of the cardiac contractions rapidly diminishes during this rise, and before the frequency of the respirations has been affected. The second, and larger, dose did not influence the arterial tension so powerfully, nor did it exert so immediate an action on the frequency of the heart's beats. Both doses were administered in such a manner that their absorption was comparatively slow; in the following experiments, physostigma was directly injected into the circulation, and its action was, therefore, more rapidly and energetically produced.

b. *Tension of the Arterial and Venous Systems.*

Experiment LIX.

Performed on a large retriever dog, vigorous and in perfect condition.

Time.	Time after Administration.	Arterial Tension. Mean Pressure.	Arterial Tension. Pressure Oscillates between	Venous Tension.	No. of Cardiac Contractions.	No. of Respirations.	Temperature.	Notes of Operations and of Symptoms.
H. M. S.	M. S.				Per Min.	Per Min.	°	
2 58 0	...	...	...	...	...	...	...	The right carotid artery was exposed, and attached to the hæmadynamometer.
3 1 0	...	...	...	...	...	...	...	The left jugular vein was attached to the second hæmadynamometer.
3 2 0	...	5·8	4·3 & 7·0	...	...	...	...	
3 3 0	...	6·0	5·0 7·0	2·8	...	...	...	
3 4 0	...	6·0	4·5 7·5	2·4	...	...	...	
3 5 0	...	6·0	4·6 7·0	2·2	...	...	...	
3 6 0	...	6·2	5·0 8·0	2·2	...	...	...	
3 7 0	...	7·0	5·0 8·0	1·9	...	...	...	
3 8 0	...	6·1	4·0 7·0	1·7	...	...	...	
3 8 30	...	5·5	5·0 7·0	...	...	...	...	
3 9 0	...	6·3	3·3 7·0	1·7	120	...	...	
3 10 0	...	5·9	3·0 6·7	...	...	8	99·2	
3 12 0	...	5·4	4·0 6·5	1·8	...	...	99·8	The dog seems perfectly calm.
3 13 0	...	5·9	4·0 7·0	...	126	10	...	
3 14 0	...	5·6	4·5 6·5	...	132	...	100·0	
3 15 0	...	5·6	4·5 7·0	...	...	10	...	
3 15 30	...	5·5	5·0 7·0	...	...	...	...	
3 16 0	...	6·6	5·0 7·0	1·8	...	12	99·9	
3 16 15	...	...	...	...	...	...	...	Commenced the injection of six grains of extract, in twenty minims of water, into the right jugular vein.
3 16 30	...	...	...	...	...	...	...	The injection was finished.
3 17 0	0 30	6·0	4·0 7·0	2·8	120	...	...	
3 17 30	1 0	4·3	...	...	...	...	...	The muscles are twitching; fæces passed.
3 17 35	1 5	4·9	...	...	...	...	...	
3 17 45	1 15	6·5	4·0 11·0	...	...	...	...	
3 17 50	1 20	7·0	5·0 11·5	...	...	...	...	
3 18 0	1 30	8·4	5·2 11·9	3·4	78	10	99·9	Urine passed; saliva is escaping from the mouth in large quantity.
3 18 30	2 0	8·5	5·5 12·0	...	...	...	...	
3 19 0	2 30	7·9	5·0 10·0	3·4	...	...	...	The twitches are now severe, and have a spasmodic character.
3 19 15	2 45	7·3	5·0 9·0	...	54	11	100·2	
3 19 30	3 0	6·5	4·5 8·0	3·4	...	...	...	No movement of the eyelids when the cornea or conjunctiva is touched.
3 20 0	3 30	5·4	4·0 8·0	3·4	40	...	100·4	The muscular twitches have become less marked.

Experiment LIX.—continued.

Time.	Time after Adminis-tration.	Arterial Tension. Mean Pres-sure.	Arterial Tension. Pressure Oscillates between	Venous Ten-sion.	No. of Cardiac Contrac-tions.	No. of Respira-tions.	Tem-perature	Notes of Operations and of Symptoms.
H. M. S.	M. S.				Per Min.	Per Min.	°	
3 20 15	3 45	5·0	...	...	...	...	...	Excessive lachrymation.
3 20 30	4 0	4·7	2·5 & 6·0	3·2	...	...	...	
3 20 40	4 10	3·7	3·0 6·0	...	...	...	...	
3 20 50	4 20	3·8	3·0 6·5	3·0	...	...	100·4	
3 21 0	4 30	4·3	3·0 6·0	2·9	32	...	...	Cardiac impulse quite perceptible to the touch.
3 21 15	4 45	3·5	2·0 5·0	...	...	...	100·4	The dog is now quite quiet.
3 21 30	5 0	3·0	2·0 5·0	2·8	...	...	...	Third eyelid is protruded slightly, and the eyeball is directed downwards and inwards.
3 22 0	5 30	2·6	2·0 5·5	2·4	24	8	100·3	
3 22 15	5 45	3·1	3·0 5·0	2·2	...	...	...	
3 22 30	6 0	2·5	2·0 5·0	2·1	...	...	...	Cardiac impulse can still be felt readily.
3 23 0	6 30	2·6	2·0 4·5	2·0	22	...	100·3	
3 23 30	7 0	2·7	1·5 5·0	1·9	...	...	...	Respirations are short and jerking.
3 24 0	7 30	2·5	2·0 3·5	1·8	20	10	100·3	
3 25 0	8 30	2·6	2·0 3·5	1·9	12	...	100·3	An occasional faint muscular twitch occurs.
3 26 0	9 30	2·7	2·3 3·0	1·7	16	10	100·2	Respirations are gasps merely.
3 26 15	9 45	2·5	2·0 3·5	...	...	...	...	
3 26 30	10 0	2·4	2·0 3·0	...	...	6	...	No reflex movement on severe irritation.
3 27 0	10 30	2·2	2·3 2·5	1·7	14	...	100·2	
3 27 30	11 0	1·3	2·0	1·6	...	...	100·2	
3 27 40	11 10	...	...	1·0	...	...	...	Cardiac and respiratory movements have ceased.
3 28 0	11 30	...	...	...	...	...	100·1	
3 33 0	16 30	...	...	...	...	...	100·0	
3 35 0	18 30	...	...	...	...	...	99·7	

The exposed heart was found, in the autopsy, to be perfectly quiet; and only one or two irregular non-synchronous movements could be obtained when it was stimulated. All its chambers were full of blood. Slight peristaltic movements were observed in the small intestines.

Experiment LX.

Experiment LX.

Performed on a large and very active Newfoundland-and-retriever cross-bred dog.

Time.	Time after Administration.	Arterial Tension. Mean Pressure.	Arterial Tension. Pressure Oscillates between	Venous Tension.	No. of Cardiac Contractions.	No. of Respirations.	Temperature.	Notes of Operations and of Symptoms.
H. M. S.	M. S.				Per Min.	Per Min.	°	
2 53 0	...	...	...	...	...	...	...	The arterial hæmadynamometer was connected with the right carotid artery.
2 54 0	...	...	...	...	...	...	...	The left jugular vein was attached to the venous hæmadynamometer.
2 56 0	...	6·0	4·5 & 7·0	...	118	34	98·0	
2 57 0	...	5·8	4·0 7·0	2·4	...	...	...	
2 58 0	...	6·2	5·0 7·5	2·4	...	28	100·1	
3 0 0	...	6·0	5·0 8·0	2·5	...	...	...	
3 3 0	...	6·0	5·0 8·0	2·5	...	...	99·9	
3 4 0	...	6·0	5·0 6·5	...	176	30	...	
3 5 0	...	6·1	4·8 7·0	...	...	...	100·2	
3 6 0	...	6·1	5·0 6·5	...	180	26	...	
3 7 0	...	6·3	6·0 7·0	3·0	...	...	...	
3 8 0	...	6·1	5·0 7·2	3·5	175	...	...	
3 9 0	...	6·0	4·5 7·0	...	...	...	100·1	
3 10 0	...	6·1	5·0 7·0	2·9	160	...	...	
3 10 30	...	...	...	...	...	...	...	Commenced the injection, into the right jugular vein, of seven grains of extract, in twenty-five minims of distilled water.
3 10 40	...	...	...	...	...	...	...	The injection was finished.
3 11 30	0 50	6·0	4·7 7·0	...	...	...	...	
3 12 0	1 10	5·9	4·5 6·5	3·1	108	...	...	Marked twitches occurred.
3 12 30	1 40	6·0	4·5 6·5	3·2	...	...	100·3	
3 13 0	2 50	...	5·0 6·0	3·3	...	25	...	
3 14 0	3 20	5·9	6·0 7·0	3·2	...	...	100·3	
3 14 30	3 50	...	...	3·4	...	...	...	
3 15 0	4 20	5·8	...	4·2	...	22	100·4	
3 16 0	5 20	7·0	5·5 8·0	4·3	42	25	...	Respirations are feeble and short.
3 16 30	5 40	6·5	5·0 8·5	3·8	...	...	100·3	
3 17 0	6 20	6·8	5·5 7·5	3·2	32	...	...	Twitching interferes with, and prevents the counting of, the respirations.
3 18 0	7 20	7·6	6·7 8·9	3·5	32	...	100·5	
3 18 30	7 50	5·0	4·0 5·0	4·2	...	...	...	
3 18 45	8 5	...	...	4·0	...	...	...	
3 19 0	8 20	4·5	3·0 4·0	3·8	24	...	100·5	
3 19 30	8 50	4·0	3·0 4·5	...	...	...	...	Occasional gasps.
3 19 45	9 5	3·1	2·5 3·5	...	...	...	100·3	
3 20 0	9 20	2·6	2·0 2·8	3·5	20	10	...	
3 20 30	9 50	2·4	2·0 2·7	3·2	...	...	100·2	
3 21 0	10 20	2·3	2·0 2·5	2·9	16	13	100·2	
3 21 30	10 50	2·4	2·0 2·4	2·7	...	10	...	Muscles are now *quivering*, not twitching.

Experiment LX.—continued.

Time.	Time after Adminis-tration.	Arterial Tension.		Venous Ten-sion.	No. of Cardiac Contrac-tions.	No. of Respira-tions.	Tem-perature.	Notes of Operations and of Symptoms.
		Mean Pres-sure.	Pressure Oscillates between					
H. M. S.	M. S.				Per Min.	Per Min.	°	
3 22 0	11 20	1·9	1·5 & 2·3	2·7	8	...	100·1	
3 22 30	11 50	...	...	2·4	6	...	...	No respiratory movements, except an occasional jerking inspiration.
3 23 0	12 20	2·0	2 0 2·1	2·3	...	...	...	
3 23 30	12 50	1·8	1·5 1·7	2·1	...	...	...	
3 24 0	13 20	1·5	1·5 1·6	2·0	...	...	...	
3 24 15	13 35	...	...	1·9	...	...	...	
3 24 30	13 50	1·5	0·6 1·0	1·0	...	...	...	
3 24 45	14 5	0	0	0	0	0	...	The three indicating columns of mercury have subsided into their reservoirs.
3 25 0	14 20	...	...	...	...	...	100·0	
3 26 0	15 20	...	...	...	...	...	99·0	
3 27 0	16 20	...	...	...	...	...	99·3	

The abdomen and chest were immediately opened: the heart was dilated, full and motionless; and no peristalsis could be observed in the intestines. The diastaltic function of the cord was completely abolished, while motor nerve-conductivity was retained, for at least five minutes after death, in the sciatic, phrenic and intercostal nerves.

The principal results of these experiments are indicated so clearly that it is almost superfluous to point them out. During the first stage, the arterial tension diminishes slightly, the venous tension increases and the cardiac contractions rapidly diminish. The frequency of the respirations was increased in only the first experiment. After this, the arterial tension increases, soon arrives at a maximum considerably above its average before the poisoning, and then slowly diminishes; while the venous tension arrives at a high maximum rather later, and by more gradual stages, than the arterial, and in the same gradual manner declines until death. In neither system is the highest point reached before a very considerable fall has been caused in the frequency of the heart's contractions. The temperature rises during the poisoning, and attains its maximum near the time that the blood-pressures have commenced finally to diminish.

As these marked changes in the circulation could be produced before the respiratory function was modified to any important extent, it did not seem to me at all necessary to repeat the experiments with the addition of artificial respiration.

It is interesting to observe the same phenomena produced by physostigma after the division of the vagi nerves.

c. *Tension of the Arterial and Venous Systems after Division of the Vagi Nerves.*

Experiment LXI.

Performed on a young, full-grown and active retriever dog.

Time.	Time after administration.	Arterial Tension. Mean Pressure.	Arterial Tension. Pressure Oscillates between		Venous Tension.	No of Cardiac Contractions.	No. of Respirations.	Temperature.	Notes of Operations and of Symptoms.
H. M. S.	M. S.					Per Min.	Per Min.	°	
2 38 0	...	...	...		...	...	...	...	The carotid artery, external jugular vein and vagus nerve of each side were exposed.
2 48 0	...	...	...		...	...	...	...	The venous hæmadynamometer was attached to the left jugular vein.
2 49 0	...	...	...		1·9	118	...	99·8	
2 50 0	...	...	...		1·8	...	...	99·2	The right carotid artery was connected with the second hæmadynamometer.
2 51 0	...	...	...		2 0	...	18	98·8	
2 52 0	...	4·9	4·0 &	6·0	...	102	...	98·7	
2 53 0	...	4·7	3·5	5·5	1·5	...	20	99·2	
2 54 0	...	5·0	4·0	6·0	2·2	108	...	99·0	
2 55 0	...	4·2	3·5	5·0	...	...	...	99·3	
2 56 0	...	...	...		...	80	...	99·2	
2 57 0	...	...	...		...	92	...	99·2	
2 58 0	...	4·5	3·0	5·5	2·1	...	...	...	
2 59 0	...	4·4	3·0	5·0	2·1	...	..	...	
3 0 0	...	4·6	3·5	5·5	2·1	90	16	99·2	
3 1 0	...	...	...		...	...	...	99·3	The right vagus nerve was cut through.
3 1 15	...	...	...		...	...	...	99·3	The left vagus was cut through. The respirations became short and spasmodic.
3 2 0	...	...	...		2·5	60	...	99·4	
3 3 0	...	6·0	5·5	6·0	2·6	...	...	99·3	The mercury in the oscillating column moves very rapidly over short spaces.
3 3 30	...	5·8	5·5	6·2	...	...	...	...	
3 4 0	...	5·8	5·8	6·2	2·5	120	...	99·2	
3 4 30	...	5·9	5·5	6·0	2·5	..	...	...	
3 5 0	...	5 9	5·5	6·0	2·5	125	...	99·2	
3 5 30	...	5·9	5·3	5·8	...	140	...	...	
3 6 0	...	5·8	5·5	5·8	2·5	170	30	99·1	
3 6 30	...	5·7	5·3	5·7	...	166	...	...	
3 7 10	...	5·7	5·7	6·1	...	166	...	99·2	Commenced to inject slowly, into the right jugular vein, seven grains of extract in twenty-five minims of water.
3 7 30	...	...	...		...	...	...	...	Finished the injection.
3 8 0	0 30	5·6	5·0	5·5	...	79	20	99·4	
3 8 30	1 0	5·6	5·0	8·0	...	...	...	...	
3 9 0	1 30	...	4·0	7·0	2·8	50	...	99·3	Respiration spasmodic ; great salivation.

Experiment LXI.—continued.

Time.	Time after administration.	Arterial Tension. Mean Pressure.	Arterial Tension. Pressure Oscillates between	Venous Tension.	No. of Cardiac Contractions.	No. of Respirations.	Temperature.	Notes of Operations and of Symptoms.
H. M. S.	M. S.				Per Min.	Per Min.	°	
3 9 30	2 0	5·4	4·5 & 7·5	3·2	...	...	99·3	Slight twitches have commenced.
3 9 40	2 10	...	...	3·7	...	...	...	Tears are flowing copiously.
3 10 0	2 30	6·8	5·0 9·0	4·2	45	...	98·5	The respirations cannot be counted, because of the twitching.
3 10 15	2 45	6·9	5·0 9·5	4·5	...	...	...	Fæces passed.
3 10 30	3 0	7·3	5·0 9·5	4·9	...	...	...	Tremors are very strong.
3 10 45	3 15	...	...	5·2	...	...	...	
3 11 0	3 30	6·8	4·5 8·0	5·2	...	...	98·2	
3 11 15	3 45	6·8	4·5 7·5	5·1	...	...	...	
3 11 30	4 0	7·8	6·5 9·5	5·0	...	...	...	
3 11 45	4 15	...	...	5·2	...	...	98·0	
3 12 0	4 30	8·1	6·0 10·0	5·3	42	...	...	
3 12 15	4 45	7·5	6·0 8·0	5·4	...	...	...	
3 13 0	5 30	6·3	5·5 7·5	5·3	56	10	98·2	Respirations are merely gasps.
3 13 30	6 0	5·6	4·0 5·5	...	...	...	...	Neither cornea nor conjunctiva is sensitive.
3 13 35	6 5	4·7	4·0 5·5	5·3	...	...	98·2	Twitches have nearly ceased.
3 13 40	6 10	4·0	...	...	...	...	...	
3 13 45	6 15	3·8	3·0 4·5	5·3	...	...	...	
3 14 0	6 30	3·5	3·0 4·0	5·3	40	an occasional gasp.	98·0	
3 14 30	7 0	3·8	3·0 4·5	5·2	...	...	...	The muscular movements can scarcely be seen, but they can readily be felt.
3 14 45	7 15	3·8	3·0 5·0	4·8	25	Do.	97·7	
3 15 15	7 45	...	...	4·2	...	...	...	
3 15 30	8 0	...	...	3·4	...	...	...	
3 15 45	8 15	...	...	3·0	...	...	...	Urine passed.
3 16 0	8 30	1·5	1·0 1·5	2·4	10	...	...	
3 16 30	9 0	...	...	2·2	...	...	97·6	
3 17 0	9 30	1·0	...	2·2	...	...	...	
3 17 15	9 45	...	...	1·9	...	...	...	
3 17 30	10 0	...	...	1·5	...	...	97·5	
3 17 45	10 15	...	...	1·3	...	...	...	
3 18 0	10 30	...	...	1·0	...	...	97·2	
3 18 30	11 0	..	...	...	...	...	...	
3 20 0	12 30	...	...	...	...	...	97·0	
3 22 0	14 30	...	...	...	...	...	96·8	
3 24 0	16 30	...	...	...	...	...	96·5	

The autopsy was made immediately. Slight, but distinct, peristalsis was observed in the intestines. The heart was dilated and motionless, and its right side contained more blood than in the preceding experiments. The phrenic, intercostal and sciatic nerves were active.

It is, therefore, shown that the same effects are produced on the circulation after the division of the vagi, as when these inhibitory nerves retain their connec-

tion with the heart. The action on the frequency of the heart's beats is well illustrated in this experiment; from their number having been considerably increased before the exhibition of the poison, by the division of the vagi nerves.

The distance over which the mercury travels in the oscillating column seems to be increased as the effects of the poison manifest themselves. This appears from all the experiments in this series, but especially from the second and the last. The division of the vagi, in the last experiment, had abnormally diminished the distance of oscillation; and, yet, it became much greater after the poisoning than it had been previous to the nerve-division. It further appears that this increase in the oscillating distance occurs when the arterial tension is about its maximum. It can only be explained by a very decided increase in the force of the cardiac contractions. I believe that this effect on the heart is altogether a reflex one, due to the resistance to the propulsion of the blood, that the augmentation in the general vascular tension must excite. In a normal condition, a stimulus of this nature might be expected to operate by increasing the *number* and not the *strength* of the cardiac contractions; but any tendency to increased frequency is opposed by the action of physostigma, for we have already seen that this substance diminishes the number of the contractions, by prolonging the diastolic pause. Their strength may, however, continue unchanged; and, during the operation of physostigma, a stimulus may even increase it without affecting the number of the beats, so long as the ganglia that initiate the systolic contraction have their excitability merely lowered without being destroyed and the contractile power of the cardiac muscle continues undiminished.

Before discussing any further the changes of blood tension, it will be advisable to examine the condition of the minute blood-vessels during the action of physostigma.

2. *Examination of the Calibre-Changes in the smaller Blood-Vessels.*

The facility with which frequent measurements may be made of the diameter of any selected capillary or minute artery or vein in the web of the frog's foot, is well known to physiologists, and has been taken advantage of by WHARTON JONES,* BENNETT,† LISTER‡ and others, in examining their condition during inflammation. A more general application of such examination of the capillaries to the investigation of the action of poisons would certainly be of great value. It has proved so in the case of nicotia;§ and we cannot consider any research on the actions of a cardiac poison to be complete unless it be done.

* On the State of the Blood and Blood-Vessels in Inflammation, &c. GUY's Hospital Reports, 1851.

† Principles and Practice of Medicine.

‡ Philosophical Transactions, 1858.

§ CLAUDE BERNARD, Leçons, &c., 1857, p. 399.

In my first experiments, the frog was merely tied down in the manner usual when the circulation in its web is being microscopically examined, but it was found impossible to prevent movements so absolutely as was required to retain the selected vessels in the field of the microscope. I ultimately found it necessary to adopt LISTER's recommendation of dividing the spinal cord several hours before the observations were begun.

Experiment LXII.

I divided the spinal cord at the occiput of a light-coloured frog weighing 529 grains, and, six hours afterwards, placed the web of one of its feet on the stage of a microscope. A small artery, a minute branch and a vein were selected and placed conveniently for measurement.*

A = diameter of larger artery; B = diameter of smaller artery; and C = diameter of vein.

Time.	A.	B.	C.	Notes.
15 minutes before administration of Calabar bean	8·0	2·5	6·5	
10 Do.	8·0	2·5	6·5	Circulation free.
5 Do.	8·0	2·5	6·5	
Time after administration of poison.				Four grains of extract, in twenty minims of distilled water, were injected into the subcutaneous tissue of the abdomen.
6 minutes,	7·8	2·5	6·5	
9 ,,	6·5	2·0	6·0	
10 ,,	5·9	1·8	5·7	
12 ,,	6·8	2·0	6·2	Circulation feeble, oscillating sometimes; vessels are crowded.
14 ,,	7·0	2·0	6·5	
15 ,,	8·5	3·0	6·5	
17 ,,	8·5	3·0	7·0	Almost no circulation.
19 ,,	9·0	3·0	7·5	Faint oscillations only in the artery; considerable crowding in all the vessels.
23 ,,	8·5	3·0	7·5	
29 ,,	8·5	3·0	7·5	Complete stasis.
36 ,,	8·5	3·0	7·5	
39 ,,	9·0	3·0	7·5	
1 hour	9·0	3·0	7·5	
2 hours	9·0	3·0	7·5	

Experiment LXIII.

The web of a frog, of 590 grains weight, was placed in the field of a microscope, after the animal had been prepared in the manner described in the previous experiment.

* NACHET's eye-piece No. 1, and object-glass No. 3, were employed; and the measurements represent divisions of an eye-piece micrometer, each of which equals $\frac{1}{2000}$th of an inch with the above glasses.

A = diameter of a small artery; B = diameter of a very small vein.

Time.	A.	B.	Notes.
15 minutes before poisoning,	5·0	3·0	
10 Do.	5 0	3·0	
5 Do.	5·0	3·0	
Time after Administration of Poison.			Two grains of extract, in fifteen minims of water, were injected into the left flank.
5 minutes,	5 0	3 0	
10 "	4 7	3·0	Circulation oscillating.
15 "	4 2	3 0	
20 "	4·2	2·9	
25 "	4·0	2·9	
28 "	4·9	3·0	
29 "	5·7	3·0	
30 "	6·0	3 2	Stasis, and crowded vessels.
35 "	7·0	3·5	Slow circulation.
40 "	7·0	3·5	
45 "	8·0	3·5	Faint oscillations.
50 "	8·0	3·5	
1 hour,	8 0	3·5	
2 hours,	8·0	3·5	

The action of Calabar bean on the minute blood-vessels of the frog's web is, therefore, to contract them considerably first, and then dilate them. The contraction may be influenced, to a slight extent, by the reduction in the frequency of the heart's action; but the succeeding dilatation, during a still greater reduction, would lead us to suppose that it is mainly due to a specific effect on the ganglia and nerves that govern the calibre-changes of the vascular system, because it is by their influence that the final dilatation must be produced.

We are now in a position to explain the changes of blood tension that have been described in mammalians. The slight fall that usually occurs in the mean pressure immediately after the poison has been exhibited, I believe to be solely due to the diminution in the rate of the heart's contractions, which has always been caused by that time. The subsequent rise in both arterial and venous tensions before any considerable embarrassment of the respiration, may be satisfactorily explained by such contraction of the smaller arteries and veins as has been demonstrated to occur in the vessels of the frog's web. It cannot be caused by increased cardiac pressure; for the heart is at the time contracting with only one half its normal rapidity, or with even less; while the greater force of each heart beat the increased oscillating distances appear to indicate, is quite insufficient to account for the high degree of blood tension sometimes attained. The subsequent, more or less rapid diminution of pressure in both arterial and venous systems is the evident result of the great dilatation in the minute blood-vessels, assisted by the weakening of the *vis e tergo* that this poison quickly produces.

Action on the Temperature of the Body.

The temperature of mammalians that are being poisoned by Calabar bean rises slightly, so that a thermometer, placed either in the subcutaneous tissue or in the rectum, will indicate a gradual elevation as soon as well-marked symptoms begin to be produced. The effect on the surface temperature has been already shown in Experiments LIX., LX. and LXI.; that on the internal temperature was frequently observed with rabbits, and may be illustrated sufficiently in the following experiment.

Experiment LXIV.

A full-grown rabbit was placed on its back on a board, and firmly secured by a ligature round each leg.

Time. H.	Time. M.	Time after Administration of Poison. M.	Time after Administration of Poison. S.	Temperature. °	Respirations. Per Min.	Notes.
12	5	...		...	...	The bulb of a delicate thermometer, with Fahrenheit's scale, was secured in the rectum.
12	10	...		99·0	...	
12	20	...		98 5	...	
12	30	...		98·4	...	
12	40	...		98·2	...	
12	50	...		98·0	47	
1	0	...		97·5	...	
1	10	...		97·3	...	
1	20	...		97·0	48	
1	30	...		96·5	...	
2	0	...		96·0	48	
2	10	...		95 8	...	
2	20	...		95·7	48	
2	30	...		95·5	48	
2	40	...		95 5	48	
2	50	...		95·3*	48	
2	56	...		...	...	Half a grain of extract, in ten minims of water, was injected into the subcutaneous tissue of the left flank.
3	1	5	0	95·8	...	Faint tremors of legs and of head.
3	5	9	0	96·3	...	Tremors are now general and stronger; and they prevent accurate determination of the respiratory movements.
3	6	10	0	96·5	...	
3	8	12	0	96·0	...	
3	15	17	0	95·5	...	
3	16	18	0	95·3	72	The muscular tremors have greatly diminished. Respirations are shallow and gasping, and they appear much impeded by bronchial secretion.
3	25	29	0	95·0	...	Quiet; except a few occasional twitches.
3	30	34	0	95·0	24	Respirations are very noisy and laboured.
3	32	36	0	94 5	...	The animal is dead.
3	35	39	0	94·0	...	
3	50	54	0	93·0	...	

* It will be observed that the temperature has gradually fallen in this experiment before the

The inconsiderable elevation of temperature exhibited in these experiments is probably an effect of the general muscular twitching that Calabar bean causes in mammals. It is quite possible that it may be also in part the result of vascular dilatation: but I think the first explanation is sufficient alone; and this additional one seems improbable when we remember that during the dilatation the circulation is extremely sluggish, because of the great diminution in the number of the cardiac contractions that accompanies it.

Action on the Blood.

The blood obtained from animals that have been poisoned by Calabar bean is generally dark in colour, because of the usual cause of death; but if drawn from the left side of the heart after a very large dose of the poison, it has the scarlet hue of arterial blood. It frequently remains semifluid for some time, and then clots loosely. When examined with the spectroscope, no modification has ever been observed in the characters or positions of the normal crurine bands.

In dogs and rabbits, the red blood corpuscles are changed in form, and present various irregularities of outline, among which a well-marked stellar crenation predominates. There can be no doubt that this is an effect produced by physostigma, as I have frequently examined the blood previous to the administration, and found its microscopic characters perfectly normal, and repeated the examination immediately after death, and invariably observed the above modifications. No change is produced in the red corpuscles of birds or frogs, nor in the white corpuscles of any animal I have examined.

For the purpose of detecting any possible effect on the respiratory function of the blood—a subject to which Harley, by his elaborate researches, has directed considerable attention*—two experiments were performed, the results of which agreed very closely. One of these may be given here, but without the numerous details which are necessarily connected with it.

Experiment LXV.

A small quantity of blood was directly removed from the right side of the heart of a Skye-terrier dog, by passing a gum-elastic catheter down the right jugular vein, affixing a syringe with a stop-cock, and withdrawing the requisite amount, according to the ingenious method of Claude Bernard.† The catheter was then detached from the syringe, and, the stop-cock

poison was administered, and apparently because of the constrained position in which it was necessary to retain the rabbit. In other nine experiments of the same description, a similar fall occurred. I endeavoured to find if a stationary, constant point could be obtained, after which the poison might be given; with the following result:—When the thermometer was introduced, the temperature was 97°; in one hour, it had fallen to 96°·3; in two hours, to 95°·6; in three hours, to 95°·3; in four hours, to 94°·7; in five hours, to 93°·3; and in six hours, to 92°. The rabbit was now set free: it was unable to stand; and other four hours afterwards, it was found dead. As this is a very ordinary method of treating rabbits during physiological experiments, it is important to recognise this injury to their vitality, which may occasion many fallacious conclusions if overlooked.

* On the Influence of Physical and Chemical Agents upon Blood; with special reference to the mutual action of the Blood and the Respiratory Gases. Phil. Trans. 1865, p. 687.

† Action de l'Oxyde de Carbon sur le Sang, Leçons, &c., 1857, p. 166.

having previously been closed, a bent steel tube was substituted by which means the blood could be readily passed into a small absorption tube over mercury, without coming in contact with the atmosphere.

The vein was now ligatured, and the dog was poisoned with a moderate dose of the extract. At the moment of death, a second quantity of blood was withdrawn with the same precautions as the first, and it was placed in a similar absorption tube. A nearly equal portion of atmospheric air was added to each tube, and they were frequently shaken during twenty four hours in a room with a temperature that varied little from 50° F. The gases were then removed and analysed. The following are the results :—

Gases from Blood before Poisoning.		Gases from Blood after Poisoning.	
Volume of blood = 7·77 cub. cent.		Volume of blood = 4·03 cub. cent.	
„ air = 47·95 „		„ air = 48·20 „	
After contact with the blood for 24 hours, air measures 51·65 c. c.		After contact with the blood for 24 hours, air measures 50·54 c. c.	
∴ Apparent exhalation = 3·70 c. c.		∴ Apparent exhalation = 2·34 c. c.	
Composition per cent. :—		Composition per cent. :—	
Oxygen, . . .	20·63	Oxygen, . . .	19·93
Nitrogen, . . .	77·82	Nitrogen, . . .	78·77
Carbonic acid, . .	1·55	Carbonic acid, . .	1·30
	100·		100·

These results agree sufficiently to prove that the respiratory function of the blood is not interfered with in physostigma poisoning. Had there been any marked discrepancy, a suspicion of such an action might be raised; but I doubt if this could be really settled without a much more refined method of experiment than was adopted. HARLEY's results seem open to very many objections, as his usual method permitted of an even greater number of fallacies than were possible in the two experiments I performed. At the same time, his paper is an extremely valuable and elaborate one, and contains many conclusions of the highest interest to physiologists.

ACTION ON THE LYMPH-HEARTS OF THE FROG.

The lymph-hearts discovered by MÜLLER* and PANIZZA† in amphibia, have always been found paralysed at an early stage of the poisoning, in the experiments where their condition was examined. As the pulsations of the pair situated one on each side of the sacrum of frogs may be readily determined without any operation, attention was especially directed to them. The time at which they cease to contract is noted in the following experiment; and the previous increase in rapidity that is there mentioned has been observed on other occasions.

Experiment LXVI.

The lymphatic hearts in the ischiadic region of a frog had an average rate of forty-nine contractions in the minute. Five minutes after two grains of extract had been subcutaneously

* Philosophical Transactions, 1833, p. 559.

† Proceedings of the Royal Society of London, vol. ix. p. 559.

administered, these contractions were seventy-two; in ten minutes, they were seventy; in fifteen minutes, they were sixty-two; and in twenty minutes, they were forty-four. At this time, the frog was flaccid, though still possessing the power of feeble voluntary movement; and its respiration had ceased. One minute thereafter, or twenty-one after the poison had been given, the most careful examination failed to detect any pulsation of the lymphatic hearts, their final stoppage having suddenly occurred.

In many other experiments, these symptoms merely repeated themselves.

Action on the Peristaltic Movements of the Abdominal Viscera.

A close analogy exists between the action on the minute blood-vessels and that on the peristaltic movements of the intestines. In mammalians, physostigma seems, in the first place, to increase the vermicular contractions of all the abdominal viscera, and then to diminish them. For some time, the intestines move with increased vigour; they then contract, so as very considerably to diminish their calibre; and, finally, they assume a condition of dilatation with lessened movement. Peristalsis has been invariably observed to continue after death; but if a large dose has been exhibited, it may be very slight and of short duration. Stimulation of the vagi nerves sometimes increases vermicular movements after death, and it then does so very conspicuously in the stomach. I have never succeeded in convincing myself of the activity of the splanchnic nerves in post mortem examinations; but, as the intestinal movements are then usually sluggish, it is a matter of extreme difficulty to judge of the action of their inhibitory nerves. During the progress of the symptoms in rabbits, I have also observed very energetic peristalsis in the cornua and body of the uterus, and even in the ureters. From the former of these effects I should be inclined to recommend physostigma as an oxytocic. As with those of the heart, the special ganglia of the intestines appear to have their functions retained, with diminished activity, for a considerable period after death; and, as with the cardiac muscle, stimulation produces non-peristaltic movements of those abdominal viscera that possess a muscular structure, long after the nerves that govern their rhythmical contractions have been paralysed.

Action on the Pupil.

In many experiments, the condition of the pupil was carefully observed, and its diameter was measured at intervals by means of a graduated glass scale, each division of which represented one-fiftieth of an inch.

The changes that usually occur in mammals may be briefly described as consisting of a short period of slight dilatation and a succeeding one of contraction; and either the latter gradually increases until death, or the pupil first oscillates once or twice between dilatation and contraction. The latter condition is present at death, and after this the pupil again dilates.

These iridal movements are best seen in rabbits; they are sluggish in dogs, more so in certain birds, and least evident in frogs.

The more important of the results may be conveniently arranged in the form of a table.

Table of Pupil-Changes during Poisoning by Physostigma.

Experiment.	Animals.	Average before Poisoning.	Minimum after Poisoning, and Time of Occurrence.	Time of Return to Average before Poisoning.†	Dilatation over Average, and Time of its Occurrence.	
No. LXVII.	Frog	3 × 4*	1·75 × 3 in 46 min.	Not noted	None noted	
„ LXVIII.	Do.	5 × 7	4· × 5 in 29 „	4 hours	Do.	
„ LXIX.	Do.	5 × 7	4·5 × 6 in 16 „	55 min.	6 × 7 in 1 h. 10 m.	
„ LXX.	Rabbit	13 long diam.	4 in 7 min.	Not noted	None noted	
„ LXXI.	Do.	9 „	4 in 5 „	15 min.	Do.	
„ LXXII.	Do.	10 „	7 in 11 „	23 min.	Do.	In 23 minutes, when the pupil had returned to its average, the animal had nearly recovered from a small dose; the second line marks the effects produced by a second and larger dose, 24 minutes after the first.
„ LXXII.	Same rabbit	10 „	6 in 7 „	Not noted	Do.	
„ LXXIII.	Rabbit	10 „	3 in 12 „	Do.	Do.	
„ LXXIV.	Do.	11 „	3 in 16 „	58 min.	Do.	
„ LXXV.	Do.	9 „	3 in 7 „	8 min.	Do.	
„ LXXVI.	Do.	12 „	4 in 8 „	Not noted	Do.	
„ LXXVII.	Do.	13 „	3 in 8 „	Do.	Do.	
„ LXXVIII.	Do.	11 „	3 in 16 min. 30 sec.	17 min. 30 sec.	Do.	
„ LXXIX.	Frog	4 × 6	3·5 × 5 in 24 min.	Not noted	Do.	
„ LXXX.	Do.	5 × 6	4· × 5 in 25 „	2 hours	5 × 6 in 1 h. 46 m.	Eyelids closed, so that it was difficult to see the pupil after 24 minutes. This often happens in frogs.
„ LXXXI.	Do.	5 × 6	3· × 4·5 in 16 „	2 hours	None noted	
„ LXXXII.	Do.	5 × 6	4· × 5 in 30 „	1 hour 47 min.	Do.	
„ LXXXIII.	Dog	21	17 in 28 min.	29 min.	25 in 30 min.	
„ LXXXIV.	Frog	5 × 6·5	4· × 5 in 19 min.	2 hours 5 min.	None noted	
„ LXXXV.	Do.	4 × 5·5	3·5 × 5 in 11 „	22 min.	Do.	
„ LXXXVI.	Do.	6 × 7·5	3· × 5 in 50 „	Not noted	Do.	
„ LXXXVII.	Dog	21	15 in 4 min.	4 min. 30 sec.	25 in 5 min. 30 sec.	
„ LXXXVIII.	Do.	10	4 in 5min. 30 sec.	8 min.	15 in 10 min.	
„ LXXXIX.	Do.	16	3 in 7 min.	9 min. 30 sec.	None before death	
„ XC.	Do.	15	9 in 11 „	16 min.	None noted	
„ XCI.	Pigeon	6	4 in 8 „	Not noted	None noted	
„ XCII.	Do.	6	4 in 28 „	Do.	Do.	General symptoms were very slowly produced.

* The figures represent fiftieths of an inch. † In warm-blooded animals, the periods entered in this column are generally subsequent to death.

In many other experiments, the pupils are described as having contracted during the poisoning, but the exact changes were not measured. In the Table, Experiments LXXV., LXXVIII., LXXXIII., LXXXVII., LXXXVIII. and LXXXIX. illustrate the rapid change that frequently occurs from contraction to dilatation; and it is obvious that unless special and continued attention be directed to the condition of the pupils, the contracted state will frequently escape detection. HARLEY,* AMÉDÉE VÉE,† NUNNELEY,‡ LASCHKEWICH§ and VAN HASSELT ‖ agree with me in describing contraction of the pupils as one of the effects that follow the internal administration of Calabar bean.

It is, unfortunately, impossible to enter fully into the question of the method in which physostigma produces its effects on the pupil, as the physiology of pupillary changes is yet unsettled, and as even the structural anatomy of the iris is a subject of debate. I am anxious to avoid being committed to any theoretical assertion on this subject, especially as the opinions I previously expressed do not seem so certainly supported by my further experience as to permit of their reassertion.

Many endeavours have been made to arrive at some definite conclusion, and, although this has not yet been attained, as in the state of our knowledge of the anatomy and normal physiology of the iris it could not be, the probable method of action may be indicated with the aid of the following experiment.

Experiment XCIII.

The two sympathetic nerves were exposed at the neck of a white rabbit. Both pupils had a diameter of seven-fiftieths of an inch. The left sympathetic was divided.

In 2 minutes, left pupil = 5, right = 7.

I then endeavoured to fix definitely the strength of the weakest interrupted galvanic current that could so stimulate the portion of the divided sympathetic next to the eye as to produce dilatation of the pupil. For this purpose, DANIELL'S cell and DU BOIS REYMOND'S induction apparatus were employed.

When the secondary coil was at 500, the resulting current produced no effect on the pupil in 30 sec.

"	480,	"	"
"	400,	"	"
"	300,	"	"
"	250,	"	"
"	200,	"	"
"	150, the left pupil dilated from 5 to 15, immediately.		
"	190, the resulting current produced no effect on the pupil in 30 sec.		
"	185, the left pupil dilated from 6 to 15 in 10 seconds.		

One grain of extract, in fifteen minims of distilled water, was injected into the subcutaneous tissue of the right flank. In thirty seconds, tremors occurred; and the symptoms rapidly advanced to a fatal termination, fifteen minutes and thirty seconds after the administration of the poison.

* *Op. cit.* p. 140. † Recherches sur la Fève du Calabar, 1865, p. 22, &c.

‡ *Op. cit.* p. 12. § *Op. cit.* p. 300.

‖ Mentioned by DONDERS (Accommodation and Refraction of the Eye: New Sydenham Society, 1864) as having been observed in 1856; and, I am informed by Professor DONDERS, communicated to a scientific society, but not otherwise published by VAN HASSELT.

The examination of the cervical sympathetics was resumed soon after the poison was exhibited, with the following results :—

In 2 min.	after the administration, stimulation of the left sympathetic with the secondary coil at		190=0 in 30 seconds.
4 „	do.	do.	185=dilatation from 7 to 15 in 10 sec.
10 „	do.	do.	188=0 in 30 seconds.
10 „ 30 sec.	do.	do.	185=dilatation from 5 to 15 in 10 sec.
12 „ 30" „	after the administration, left pupil = 4, right pupil = 5.		
13 „ 30 „	do. stimulation of the left sympathetic with the secondary coil at		185=0 in 30 seconds.
14 „ 0 „	after the administration, stimulation of the right sympathetic with the secondary coil at		185=0 „
14 „ 35 „	after the administration, stimulation of the left sympathetic with the secondary coil at		160=0 „
15 „ 30 „	do.	do.	90=0 „

The rabbit was now dead, and both pupils had been for more than two minutes previously in a state of extreme contraction.

In 16 min. 10 sec.	after the administration, stimulation of the right sympathetic with the secondary coil at	30 = 0 in 30 seconds.
17 „ 0 „	after the administration, stimulation of the left sympathetic with the secondary coil at	10 = 0 „
18 „ 0 „	after the administration, stimulation of the right sympathetic with the secondary coil at	10 = 0 „

The left iris was at this time exposed by cutting away a portion of the cornea. The electrodes were applied directly to its surface, near the external margin, when a slight and rapid contraction of the iris (dilatation of the pupil) occurred, instantly followed by a rebound to its previous condition. This effect was frequently produced during many minutes after death, but no distinct expansion of the iris (contraction of the pupil) could be caused when the electrodes were applied in the same way to the pupillary margin. It must, however, be added that the pupil was at this time in a very contracted condition.

We learn from this experiment that the cervical sympathetic is paralysed before the death of the animal, while a portion of the apparatus that is immediately concerned in the contraction of the iris retains its vitality for a considerable period afterwards. The cause of the pupillary contraction during poisoning by internal administration is, therefore, in all probability, to be found among those consequences that naturally succeed the removal of the influence of the cervical sympathetic nerve. Without dogmatising on this subject, and feeling content in the meantime with the mere narration of these facts, I am inclined to think that such changes of the iris can only be explained by considering the influences of dilator and constrictor muscles, and also of a system of contractile blood-vessels.

An animal struggles violently during the action of a poison, and the pupil dilates. It is natural to suppose that, in this case, the excited spinal nerves had produced increased action of the dilator muscle, and that thereby the antagonism of the constrictor had been overcome. But if iridal movements are merely the results of spinal or cerebral nerve-force interfering with antagonism between muscles, how account for the continuation of either dilatation or contraction after

the death of these nerves? Such conditions should not exist when the causes of interference with antagonism have been removed. I, therefore, venture (in common with others, and notwithstanding the anatomical difficulties that exist) to include an arrangement of contractile blood-vessels among the causes that produce iridal movements.

These blood-vessels will possess the function of erectile texture, and will act either in harmony with the dilator or constrictor muscles, or independently of them. The coincidence I have frequently observed between changes of blood-tension and differences in the size of the pupil also leads me to support this view. At the same time, the influence of an erectile tissue is not in itself sufficient to account for all the pupil changes; they can never be explained satisfactorily without also considering the effects of spinal and of cerebral motor nerves, operating probably on radiating and circular muscular fibres whose action is independent of such tissue. The cervical sympathetic appears to be the channel through which the nerves that originate in the cilio-spinal region pass to the iris. Stimulation of this region, or of the sympathetic nerve, produces dilatation of the pupil: and this does not interfere with the supposition of the existence of a contractile vascular network co-operating with a proper dilator muscle; for then the blood-vessels of this network, being governed by branches of the same nerves, would contract along with the dilator fibres, and the result would be a diminution in the size of the iris, and, consequently, a dilatation of the pupil. In the same way, division or paralysis of the sympathetic would result in iridal expansion; the contraction of the pupil being caused by dilatation of the blood-vessels, assisted, it may be, by the simultaneous contraction of a circular muscle.

B. TOPICAL EFFECTS.

When applied to the Nervous System.

It is obvious that when a poison is applied during life to the substance of any of the central nerve-organs, it will produce its specific action on the system in the ratio of the absorbing power of the organ, and therefore very much in proportion to the local blood supply. Other distinct effects are, however, frequently caused by the concentrated form and other peculiarities of the preparation.. The watery suspension of physostigma extract caused no peculiar symptom when applied to the cerebrum of mammalians, birds, or frogs; and as its absorption was slow when so exhibited, the constitutional effects were produced only after long periods. When it was applied to the spinal cord of frogs, peculiar twitchings occurred in the muscles directly connected by motor nerves with the part of the cord in contact with the poison. These twitches soon ceased, and no movements were then caused when this portion of the cord was galvanised. The first effect was probably the result of local irritation merely, while the final paralysis was due to a specific action of physostigma.

From the nearly complete absence of blood-vessels in the trunks of the sciatic nerves, it is possible to localise the effects of a poison to any portion of the trunk. A curious result was produced by the topical application of Calabar bean. It has been shown that this poison does not appear to paralyse the afferent nerve fibres when acting through the blood; or, at least, that under its influence the function of the motor nerves is indubitably very much sooner destroyed than that of the sensory. When, however, the poison is applied to a mixed nerve-trunk, the order wherein these effects are produced is reversed, the afferent nerves being paralysed a few minutes before the efferent. This may be shown with great distinctness if strychnia be given after the local action has continued for some time.

Experiment XCIV.

The spinal cord of a frog was divided at the occiput, the sciatic nerves were exposed, and a piece of gutta-percha parchment was placed under each nerve, so as to isolate it completely. A small pad of cotton wadding *steeped in water*, was applied to the *right* nerve, while a similar pad *steeped in a concentrated mixture of extract of physostigma and water*, was placed on the *left;* care being taken to prevent the diffusion of any of the extract beyond the parchment. At this time, a slight stimulation of either nerve, below or above the pads, caused contraction of the limb and general reflex movements.

The pads were kept moist by an occasional drop of water on the right, and one of watery extract of physostigma on the left; and they were retained in their positions for an hour and forty minutes. They were then removed, and the left leg was carefully washed with distilled water, so as to remove effectually any extract that might have been adhering to the nerve. They were both tested with galvanism, when movements of the left leg followed the application of the poles to any portion of the exposed left sciatic; but no reflected contraction occurred when the nerve was stimulated below the position that had been occupied by the pad. No change had occurred either in the afferent or the efferent conductivity of any portion of the right nerve.

A drop of solution of strychnia was now applied to the wound that had been made in the neck by dividing the cord. Seven minutes after this, very weak galvanism of any portion of the right sciatic nerve caused a spasmodic shock of all the body. The same current produced a like effect when it was applied to the left nerve above the part that had been occupied by the poisoned pad; *but when it was applied to the poisoned part*, or *lower down*, the muscles below the point stimulated alone contracted, *no reflected movements being caused.* The same effects were repeatedly observed for other six minutes, before the lapse of which time the action of the strychnia had manifested itself more violently, and rendered the above peculiarities more distinct and exaggerated.

This experiment has been several times repeated, and has always yielded similar results.

If the poisoned pad be permitted to remain in contact with the nerve for a few minutes after the paralysis of its afferent fibres has been caused, the motor fibres also will have their conductivity destroyed.

Experiment XCV.

The sciatics of a frog were exposed, and were treated exactly as in the previous experiment. The afferent conductivity of that portion of nerve to which physostigma had been applied was lost in two hours, but the motor conductivity was yet retained. The application was continued, with the result that the conductivity of the motor fibres was destroyed within other fifteen minutes.

When applied to Striped and to Unstriped Muscle.

Although physostigma, when acting through the blood, does not destroy muscular contractility, the contact of a concentrated preparation is immediately followed by very rapid paralysis of the portion of muscle to which it is applied.

Experiment XCVI.

The two gastrocnemii muscles of a frog, and portions of the sciatic nerves in the thighs, were exposed and separated from contiguous structures. The muscles were completely isolated by pieces of parchment. A pad *soaked in a concentrated mixture of extract and water*, was placed on the surface of the right muscle, and a similar pad *moistened with water*, was placed on the left: the muscles being at the time completely under the control of their sciatic nerves, and being readily excited to contractions by direct galvanism.

Four minutes afterwards, a weak galvanic current was applied to the left nerve, and produced energetic contraction of all the muscles supplied by the nerve; and the left gastrocnemius muscle also contracted forcibly when directly stimulated. The same current was then applied to the right nerve, and caused pretty active movements of the right leg; in which, however, the gastrocnemius only sluggishly participated.

In fifteen minutes, the right or *poisoned gastrocnemius was perfectly paralysed; while the left muscle appeared to be as active functionally as when the experiment was commenced.*

The pad soaked in the extract was then removed from the right muscle and placed on the left. In five minutes, the contractility of the latter was considerably impaired; and in twenty minutes, no contraction could be produced, even when it was stimulated by very strong galvanism.

It was found that when a portion of intestine in energetic peristalsis, had its surface painted over with a concentrated watery mixture of the extract, it became flaccid; and that when a vermicular contraction ran along towards this portion, it stopped at the margin of the portion, and appeared to *skip over* it, as the peristalsis was resumed at the nearest unpoisoned point of the intestine. Soon afterwards, the poisoned portion of intestine could not be stimulated to contract by strong galvanism.

When applied to the Heart.

For the purpose of examining the topical effects on the heart, frogs were generally employed, but warm-blooded animals were in a few instances made use of.

The conditions of the experiments were varied by the application of the poison to the visceral pericardium, and to the muscular substance of the heart; without, and after, its removal from the body; and by the insertion of the poison into one of the cardiac chambers.

1. *Without its Removal from the Body.*

a. *To the Visceral Pericardium.*

Experiment XCVII.

The exposed heart of a frog was found, during ten minutes, to have an average rate of seventy beats in the minute.

A drop of filtered concentrated solution of the extract was placed on the pericardium.

In 1	minute, heart	= 52	per minute.	
2	„	= 40	„	
4	„	= 36	„	
6	„	= 30	„	
8	„	= 24	„	
10	„	irregular;		eighteen ventricular and twenty auricular contractions.
13	„	„		no ventricular and eleven auricular contractions.
14	„	„		„ „
22	„	= 8	per minute;	both chambers contract, but the ventricles very feebly.
25	„	= 12	„	
40	„	= 15	„	very feeble, but rhythmical.
50	„	irregular;		twelve ventricular and fifteen auricular contractions.
52	„	„		no ventricular and eighteen auricular contractions.

The ventricles did not again contract spontaneously, but the auricular action continued for many hours. Voluntary movements were made by the frog until fifty minutes after the poison had been placed on its pericardium.*

b. *To the Heart.*

Experiment XCVIII.

A portion of the pericardium was removed from the exposed heart of a frog; and, during ten minutes that followed this operation, its contractions varied little from forty-five in the minute. A small drop of the same solution of extract as was employed in the previous experiment was placed on the heart's surface.

In ten seconds, heart = 0.

It continued motionless for thirty seconds; then recommenced; but, almost immediately afterwards, the frog struggled violently, and the contractions again ceased for ten seconds.

In 2	minutes, heart	= 27	per minute.	
4	„	= 21	„	
6	„	= 30	„	
8	„	= 34	„	Rhythmical and regular.

At this time, a second drop was applied to the heart.

In 30 seconds, a struggle occurred, and the heart stopped for ten seconds. When it recommenced, the contractions were irregular, only five ventricular to ten auricular occurring in the minute.

In 3	minutes,	there were no ventricular movements and	6 auricular.		
10	„	„	20 „		
12	„	heart = 23 per minute;	rhythmical but irregular.		
20	„	„ = 28	„	„	and regular.
40	„	„ = 28	„	„	„

A third drop was now applied.

In 2	minutes after the third application,	heart irregular,		no ventricular movement, and 24 auricular.
8	„	„	„	„
18	„	„	= 7 per minute,	both ventricles and auricles contracting.
25	„	„	= 0	„

* In this and in the other experiments of the series, a drop of water was occasionally placed on the heart to prevent its surface from drying.

Experiment XCIX.

A young rabbit was killed, and its heart was exposed. During four minutes, the contractions were eighty per minute. A concentrated solution of extract was painted over the greater portion of the heart's surface. In one minute, the contractions had entirely ceased; but, a few seconds afterwards, the left ventricle spontaneously resumed its action, and in two minutes after the application, the whole heart was contracting at the rate of seventy-six per minute.

The application was thrice repeated with similar results. Latterly, however, a longer interval occurred between the suspension and recovery of the cardiac contractility. Paralysis of the heart was ultimately caused by continuing these applications.

2. *After Removal from the Body.*

When the heart is removed from the body and placed in a concentrated solution of extract, its contractions immediately become irregular, and then cease. All the vital properties of its structures are paralysed in one or two minutes.

3. *Insertion of Physostigma into one of the Heart's Chambers.*

Experiment C.

A young rabbit was killed by the destruction of the medulla oblongata. In four minutes afterwards, while the heart was contracting at the rate of fifty per minute, two grains of extract, in five minims of water, were injected by WOOD'S syringe into the right auricle. The action of the heart instantly ceased. However, during the next ten minutes, irritation could still cause single laboured contractions.

TOPICAL ACTION ON THE BLOOD-VESSELS AND PIGMENT-CELLS WHEN APPLIED TO THE WEB OF THE FROG'S FOOT.

WHARTON JONES has examined with great care the changes in the calibre of the blood-vessels that follow the application of various substances to the frog's web.* Solution of atropia produces a marked contraction, a result that I have had occasion to confirm; and it, in this respect, resembles ordinary stimuli, such as galvanism, temporary cold and heat, and various irritants. He further established, and LISTER has supported and extended the statement,† that the contractile power of minute blood-vessels is independent of the central organs of the nervous system, though it may be controlled by them. A few substances were found that cause dilatation of blood-vessels, and among them solution of opium.

This antagonism in the actions of opium and atropia, and the well-known difference in their effects on the pupil, whether acting through the blood or acting by topical application, seem to have an important bearing on the question of how far the movements of the iris are due to calibre-changes of its blood-vessels. The examination of the topical effects of physostigma—a much more powerful

* On the State of the Blood-Vessels in Inflammation, &c.; *loc. cit.*

† An Inquiry regarding the parts of the Nervous System which regulate the Contraction of the Arteries; Philosophical Transactions, vol. cxlviii., 1858.

myositic agent than opium—is thus not only of great intrinsic interest, but also likely to be of considerable importance in explaining its action on the iris. In the experiments that were undertaken for this purpose, the condition of the pigment-cells also was observed.

Experiment CI.

The spinal cord of a light-coloured, large frog was divided between the first vertebra and the occiput; and, thirty minutes afterwards, a portion of a web was placed on the stage of a microscope.

An artery and three branches were selected for examination (A, B, C and D); and during ten minutes, their diameters occupied pretty constantly the following divisions of an eye-piece micrometer :—

A = 3 ; B = 2 ; C = 2 ; D = 1·5. The pigment-cells were diffusely stellate.

A small drop of filtered watery solution of extract was placed on the web.

In 2 minutes,	A = 3·0 ; B = 2·0 ; C = 2·0 ; D = 1·5.	Pigment-cells stellate,		Current of blood seems rather more rapid.
8 „	A = 3·0 ; B = 2·5 ; C = 2·5 ; D = 1·5.	„	„	
15 „	A = 3·5 ; B = 3·0 ; C = 3·0 ; D = 1·5.	„	concentrating.	
20 „	A = 4·0 ; B = 3·0 ; C = 3·0 ; D = 1·5.	„	concentrated : no rays.	
40 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	„	
50 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	diffusing a few rays.	
1 h. 0 min.	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	rather more diffuse than at commencement of experiment.	
1 „ 10 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	in nearly extreme diffusion.	Circulation free.
1 „ 20 „	A second drop of the solution was placed on the web.			
1 „ 25 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	Pigment-cells	concentrating.	
1 „ 26 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	stellate.	
1 „ 30 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	again diffusing.	
1 „ 35 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	in extreme diffusion.	
2 „ 30 „	A = 4·0 ; B = 3·0 ; C = 3·5 ; D = 2·0.	„	„	

Experiment CII.

An artery and two branches (A, B, C) and a capillary (D) were selected in the web of frog, which was prepared as in the preceding experiment. Their diameters were found to be the following—A = 7 ; B = 6·5 ; C = 4 ; D = 1 : while the pigment-cells in the field were in a state of extreme concentration, there being no rays visible.

A drop of a filtered, strong solution of extract was placed on the web.

In 10 minutes,	A = 9 ; B = 8 ; C = 5 ; D = 1.	Pigment-cells	in extreme concentration. Circulation active.
20 „	A = 9 ; B = 8 ; C = 5 ; D = 1.	„	extreme concentration. Circulation active.
40 „	A = 9 ; B = 8 ; C = 5 ; D = 1.	„	stellate. Circulation languid.

The observations were continued until two hours after the application; and, up to that time, no further change had occurred, except that extreme diffusion had appeared in the pigment-cells.

Experiment CIII.

A portion of the web of a frog was arranged for microscopic examination, in the same way as in the two preceding experiments. An artery and two branches (A, B, C) were selected for examination, along with several adjoining pigment-cells.

Before the application, $A=4$; $B=2$; $C=1·5$: while the pigment-cells were in the stellate form.

A drop of a filtered, strong solution of extract was placed on the web.

In 5 minutes,	$A=5$; $B=2$; $C=1·5$.	Pigment-cells	concentrated, with a few short rays,	Circulation more rapid; and many capillaries that were before empty now contain an actively moving blood stream.
10 „	$A=5$; $B=2$; $C=1·5$.	„	in extreme concentration.	
20 „	$A=5$; $B=2$; $C=1·5$.	„	„	Circulation continues active.
30 „	$A=5$; $B=2$; $C=1·5$.	„	„	„

The action of physostigma on the calibre of the minute and contractile blood-vessels is thus the reverse of that of atropia.

Lister believes that the pigment-cells possess a nerve apparatus that governs their condition in exactly the same way as the smpathetic ganglia and spinal nerves govern the calibre-changes of the smaller blood-vessels.* If this be so, the long continued concentration, which precedes the diffused condition of the pigment in these cells, would imply that their nerves are more sensitive to irritation than those of the vascular system, and that this pigmentary concentration is the result of a stimulating property of the extract, of so slight a character as to have no effect on the ganglia and nerves of the blood-vessels.

When applied to the Eyeball, or to its immediate neighbourhood.

As this investigation has been confined to the lower animals, the action on the iris only can be here discussed. This has been observed in amphibia, reptiles, birds, and mammalians. The pupil contracts within a few minutes after the application of physostigma to the eyeball, to the cutaneous surface in its neighbourhood, or to the nasal mucous membrane; and, if the quantity have been considerable, this may last for two or three days. It is caused much more rapidly, and maintained for a much longer time, than the often varying state of contraction that has been described as a symptom produced by physostigma acting through the blood. There is no reason to doubt that it is as purely an effect produced by contact with the iris as the opposite state that atropia causes; and so limited is the action, that in the same animal extreme physos-

* On the Cutaneous Pigmentary System of the Frog; Philosophical Transactions, vol. cxlviii., 1858, p. 627.

tigma-myosis may exist in one eye, while atropia-mydriasis is present in the other. It is easy to prove the presence of the extract within the eyeball, after its topical application, by removing the aqueous humour and placing it on the conjunctiva of another animal, when the usual effects of Calabar bean on the pupil will be produced.

In addition to this phenomenon, increased lachrymation, and congestion of the conjunctiva, and, I believe, of the iris itself, are produced.

The question of the effects produced on the eye by the topical application of such substances as physostigma and atropia must always be an interesting one to physiologists. Its full discussion would require a special paper, so numerous are the data to be considered, and so conflicting the opinions that have been expressed. Besides, there are some effects that can only be conveniently examined in man, and which must be included in any satisfactory review of this attractive subject. It is, however, quite within the purpose of the present investigation to consider the method in which the iris is expanded by physostigma. I have already expressed an opinion that iridal changes appear to require the co-operation of special radiating and circular muscular fibres, with a system of contractile blood-vessels possessing to a certain extent the properties of erectile tissue. A mere antagonism between the two former muscular arrangements could not alone account for the effects of either physostigma or atropia; and, indeed, the actions of those substances seem to oppose such a theory. All the muscular fibres in the iris are unstriped, and physostigma relaxes while atropia contracts such fibres. Therefore, were these the only causes of iridal movement, physostigma would produce no effect on the pupil; for it would merely render less energetic the contraction of both, and would not thereby disturb their counterbalancing antagonism. For a similar reason, atropia will merely increase the opposing action of both, without causing any iridal movement.

To reconcile their actions with this anatomical arrangement, it is necessary to make one of the following very improbable suppositions—that the circular muscle at the pupillary margin has different physiological properties from the dilator; or that the one set of fibres is regulated by nervous ganglia that can be stimulated by the contact of substances, which paralyse corresponding ganglia whose power is limited to the other set.

Let it be granted, however, that dilator and constrictor muscles with counterbalancing powers exist, and that, in addition, a system of contractile blood-vessels is present, and the ascertained physiological actions of these opposing substances will be rendered available for the explanation of their peculiar effects. Dilatation of the pupil by atropia will then be due to the successful opposition of radiating muscular fibres and contractile iridal blood-vessels, to unaided constrictor fibres; the contact of atropia causing contraction of all unstriped fibres,

including those of the blood-vessels: while contraction of the pupil by physostigma will be intelligibly explained by the universal relaxation of all the contractile tissues of the iris, and the consequent enlargement of its area by the increased accession of blood, which the dilated vessels attract and permit.

The topical action of physostigma on the blood-vessels has been already described, and it certainly supports the view just stated. Confirmatory evidence is also obtained from ADAMÜK's recent experiments on intra-ocular pressure,* which show that this is increased by extract of Calabar bean applied to the conjunctiva and diminished by atropia; for such a difference of effect would necessarily exist, did physostigma augment the blood supply of the iris and atropia decrease it.

The additional effects that follow the topical application to the human eye-ball will be mentioned among the general conclusions.

The following are the conclusions of this investigation:—

A. ACTION THROUGH THE BLOOD.

1. Physostigma has proved fatal to every animal hitherto examined, with the exception of the Esĕrĕ moth. In mammals and birds, death is most rapidly caused when the poison is injected into the circulation or when it is brought into contact with a wounded surface. It follows nearly as quickly, when Calabar bean is introduced into a serous cavity; much more slowly when it is exhibited by the mucous membrane of the digestive system. In rabbits, death has been caused by its application to the Schneiderian, the auditory or the conjunctival mucous membrane. The skin of frogs resists the poison for a long time; but, if it be applied for a considerable period, and with proper precautions, distinct evidence of absorption may be obtained, though death has never been caused by such application.

2. The contact of the extract of Calabar bean with the gastric juice of a dog, for twenty-four hours and at a temperature a little above 95° F., did not, in the slightest degree, modify the energy of the poison.

3. A large dose, given to a mammal or bird, rapidly affects the cardiac contractions, and then paralyses the heart. The respiratory movements are quickly stopped, but the symptoms and post mortem appearances are those of syncope. Such a dose, injected into the abdominal cavity of a frog, affects nearly simultaneously the heart and spinal cord, and very rapidly destroys the vitality of both organs. In this case the motor nerves are only slightly, or not at all affected, and may retain their conductivity for about thirty hours. Evidence of

* Centralblatt, No. 36, 1866; and RUTHERFORD's Report on Physiology, Journal of Anatomy and Physiology, No. II., 1867.

the vitality of the afferent nerves may be obtained as long as the retained vitality of the spinal cord permits of its diastaltic function being examined.

4. In mammals and birds, an average dose produces symptoms of asphyxia. When administered to frogs, a similar dose impairs the function of the spinal cord, and diminishes the rates of the cardiac contractions and of the respiratory movements; and, soon after, the latter cease. In periods varying from one and a-half to four hours afterwards, the motor nerves are paralysed; this paralysis implicating their endorgans first, and their trunks afterwards. From this it must not be inferred that the nerve is paralysed by a centripetal progression of the poison; the only fact demonstrated being that a direct ratio exists between, on the one hand, subdivision of nerve substance, facilitating contact of the poison, and, on the other, rapidity of paralysing effect. Indeed, division of the nerve trunk, previous to the administration of Calabar bean, delayed the paralysis of the endorgans. The afferent nerves retain their activity as long, at least, as the functions of the spinal cord are not lost. The spinal cord and the motor nerves are generally paralysed at about the same time.

5. When a small, but still fatal, dose of Calabar bean is administered to a frog, the effects are the same as those in the previous conclusion, until they arrive at the stage of paralysis of the motor nerves; after this, an interval of several hours may elapse before the functions of the spinal cord are completely suspended. During this interval the *tactile* sensibility of the afferent nerves is increased: so that, if the ischiadic artery and vein of one limb have been tied before the exhibition of the poison, a slight touch of the skin in the poisoned region, which before the administration of the poison caused no effect, will now produce faint twitches of the limb whose vessels are tied; while an ordinary excitant, such as sulphuric acid, will show everywhere a marked diminution in the diastaltic activity, as measured by the métronome.

6. *With a still smaller dose, a frog may have its cardiac contractions reduced by from seventy to eight per minute, its respiratory movements completely stopped, and the endorgans of its motor nerves paralysed, and yet afterwards completely recover.* This has occurred when two grains of extract were administered to a frog, weighing 730 grains.

7. In frogs, the voluntary muscles are unaffected by the poison, and may continue to respond to galvanic stimulation during three or four days after its administration. The contrast between this and the effect of Calabar bean on the motor nerves, may be well shown by ligaturing the ischiadic vessels of one limb before injecting the poison. If, when strong stimulation causes no reflex movement, the two gastrocnemii muscles with their attached nerves be so placed that an interrupted current, from one DANIELL's cell and DU BOIS REYMOND's induction apparatus, may be transmitted simultaneously through either both muscles or both nerve trunks, it will be found in the case of the

muscles that when the secondary coil is slowly advanced contractions will occur with the same current in both muscles, or with a weaker current in the case of the poisoned than in that of the non-poisoned one, this varying with the length of time which has elapsed since the limb was deprived of blood; when the current is transmitted through both nerves, contractions will be produced simultaneously in both muscles, or with a weaker current in the non-poisoned one, or contractions will occur in the non-poisoned muscle only, this also varying with the length of time that may have elapsed since the exhibition of the poison.

8. In mammals and birds, the voluntary muscles are affected in a very remarkable manner. At an early stage of the poisoning, faint twitches occur, which gradually extend over the body, and, at the same time, increase in vigour so as to interfere with the respiratory movements. Shortly before death, they again become mere successive twitches, often requiring the use of the hand to discover their existence. After death, if a muscular surface be exposed, these twitches will still be observed, involving usually different muscular fasciculi at different times, rarely the whole of a muscle at once; and in mammals they may persist for more than thirty minutes after death. They are caused by a direct effect of physostigma on the muscular substance. This is shown by their continuing after paralysis of the motor nerves, by their persisting in a muscle cut out of the body, and by their non-occurrence in parts that have been separated by ligature from the circulation.

9. In mammals and birds, when the dose is large, the heart's action is rapidly made slower and then stopped. In dogs, it may diminish to one-half in three minutes, and cease in ten. In frogs also, a large dose, injected into the abdominal cavity, causes rapid and complete cardiac paralysis. A smaller dose causes either a gradual cessation followed by a renewal at a diminished rate, or a gradual fall, from sixty or seventy to four or six beats per minute, followed by a gradual return to a diminished rate of from eight to twenty per minute. At this stage, and for many hours afterwards, the only signs of vitality are this diminished cardiac action and the power of the voluntary muscles to respond to galvanic and other stimulation. In the frog, where alone these last phenomena have been observed, the heart may continue so to contract for three or even five days, provided the temperature of the apartment be as low as 50° F. After stoppage, galvanism may sometimes cause a renewal of rhythmical contractions; but this can rarely be done, and unrhythmical and partial contractions can alone be excited. Cessation of the heart's contractions occurs in diastole, with all the chambers full.

10. The pneumo-gastric nerves retain their inhibitory power over the heart during the whole time from the diminution to the partial recovery of its action. Soon after this, however, they are paralysed; and this occurs at nearly the same time as the affection of the motor nerves.

11. Division of the pneumo-gastric nerves, or their paralysis by curare, or destruction of the medulla oblongata or spinalis, does not protect the heart from the action of physostigma.

12. The lymphatic hearts of frogs poisoned by Calabar bean soon cease to contract.

13. In rabbits, a large dose paralyses the cervical sympathetic nerves, before the death of the animal. A smaller fatal dose merely diminishes their activity.

14. Before the stoppage of the heart, proofs may be obtained of the vitality of its sympathetic ganglia; but, as striped muscle is not affected by Calabar bean conveyed by the blood, we are obliged to infer from the symptoms respectively produced, that the activity of the cardiac sympathetic system is probably destroyed by a large dose, and lessened by a smaller one.

15. The animal temperature, both external and internal, has been invariably observed to rise in rabbits and dogs, but only slightly and for a short period; after which it slowly falls.

16. The condition of the capillary circulation was examined in the web of the frog. Soon after the exhibition of the poison, the smaller arteries and veins contracted slightly; but, after a short interval, this contraction was succeeded by a rapid and permanent dilatation, in which the calibre of the vessels was considerably above their maximum previous to the poisoning. This capillary dilatation appears to occur all over the body, as is shown by a peculiar blue coloration of the voluntary muscles and of the heart, a similar coloration of the serous and fibro-serous tissues, and a congestion of the blood-vessels in the conjunctiva and iris. This change also occurs, in a less marked manner, in birds and mammals.

17. The general results of experiments in which the arterial and venous tensions were examined were, that the arterial tension first diminished slightly, immediately after the administration of the poison, then gradually increased until it reached its maximum—when the number of cardiac contractions had diminished to at least one-half,—and afterwards rapidly fell; and that the venous tension began to increase immediately after the administration, continued doing so until it slowly reached its maximum—when the arterial tension had considerably diminished,—and then fell, though more gradually than the tension of the arterial system. The number of the cardiac contractions when the venous tension had attained its maximum, was about one-third of the average before the poisoning; the respirations were rather less frequent than before, and the temperature had risen a few tenths of a degree.

18. Physostigma causes extreme diffusion in the pigment-cells of the frog's skin, and thus a very marked change occurs in the colour of the animal during the progress of the symptoms.

19. In dogs, the peristaltic action of the intestines is usually destroyed at death; it may, however, continue a short time afterwards. In rabbits, the intes-

tinal movements are frequently increased in activity before death, and generally continue for a considerable time afterwards.

20. The pupil contracts in all cases of rapid poisoning in mammalians and birds. The contraction may, however, be slight and of short duration; and dilatation may then be observed during the greater portion of the experiment, especially if the dose be a small one. Contraction of the pupil is produced in frogs also.

21. Calabar bean acts as an excitant of the secretory system; increasing the action of the alimentary mucous, of the lachrymal, and of the salivary glands.

22. In the frog, the symptoms of poisoning are not materially altered by removal of the brain, or by division of the cervical portion of the spinal cord.

23. Artificial respiration does not prevent death, in mammals, after the exhibition of a poisonous dose. This is a necessary result of the effects of physostigma on both the cerebro-spinal and sympathetic systems.

24. Congestion of internal organs occasionally occurs; but this is by no means an invariable consequence of a fatal dose.

25. The blood is generally dark after death, but becomes arterialised on exposure to the air; its respiratory capabilities are unaltered; it often clots loosely and imperfectly; and, when examined with the spectroscope, the bands of scarlet crurine are found unchanged. In the rabbit and dog, a microscopic examination demonstrates an invariable change in the coloured corpuscles, which have their outlines distinctly crenated. This change is not caused in the blood of birds or amphibia. The white corpuscles remain unaltered.

B. TOPICAL EFFECTS.

1. When the poison is applied to the surface of a frog's brain, no effect is produced; but when it is brought into contact with the spinal cord, a few twitches occur in the extremities, followed by paralysis of the portion of cord acted upon.

2. When physostigma is applied to a mixed nerve-trunk, in a concentrated form and with proper precautions to prevent absorption by neighbouring parts, first the afferent nerve-fibres are paralysed, and afterwards the efferent.

3. Topical application destroys the contractility of striped and of unstriped muscular fibre. The heart's action is stopped by repeated application to its external surface or to the pericardium. If a small quantity be injected into one of its chambers, paralysis nearly immediately follows.

4. The blood-vessels are dilated when a solution is applied to the web of the frog's foot.

5. The effects of the application of Calabar bean to the eyeball are a somewhat painful sensation of tension in the ciliary region, contraction of the pupil, myopia and astigmatism; with, frequently, congestion of the conjunctival vessels, pain in the supra-orbital region, and twitches of the orbicularis palpebrarum muscle.

PROCEEDINGS

OF THE

STATUTORY GENERAL MEETINGS,

AND

LIST OF MEMBERS ELECTED AT THE ORDINARY MEETINGS,

SINCE JANUARY 3, 1865,

WITH

LIST OF DONATIONS TO THE LIBRARY,

FROM NOV. 28, 1864, TILL NOV. 25, 1867.

PROCEEDINGS, &c.

Monday, 28th November 1864.

At a Statutory General Meeting, His Grace the DUKE of ARGYLL, President, in the Chair, the Minutes of the Statutory Meeting of 23d November 1863 were read and confirmed.

The following Office-Bearers were elected for 1864–65:—

Principal Sir DAVID BREWSTER, K.H., LL.D., D.C.L., President.

Dr CHRISTISON,
Professor KELLAND,
Hon. Lord NEAVES,
Principal FORBES,
Professor INNES,
Professor LYON PLAYFAIR, C.B., } Vice-Presidents.

Dr JOHN HUTTON BALFOUR, General Secretary.

Dr GEORGE JAMES ALLMAN,
Professor P. GUTHRIE TAIT, } Secretaries to the Ordinary Meetings.

DAVID SMITH, Esq., Treasurer.

Dr DOUGLAS MACLAGAN, Curator of Library and Museum.

COUNCILLORS.

Dr WILLIAM ROBERTSON.
Dr E. RONALDS.
T. C. ARCHER, Esq.
W. F. SKENE, Esq.
A. KEITH JOHNSTON, Esq.
Rev. Dr STEVENSON.
Dr STEVENSON MACADAM.
Hon. Lord JERVISWOODE.
JAMES T. GIBSON-CRAIG, Esq.
EDWARD SANG, Esq.
Sir JAMES COXE, M.D.
Rev. Dr BLAIKIE.

The following List of Honorary Members was submitted before being printed in the billet of next meeting:—

I. FOREIGN.

ROBERT WILHELM BUNSEN, Heidelberg.
AUGUSTE DE LA RIVE, Geneva.
JEAN BERNARD LEON FOUCAULT, Paris.
ELIAS FRIES, Upsala.
HERMANN HELMHOLTZ, Heidelberg.
ALBERT KOLLIKER, Wurzburg.
RICHARD LEPSIUS, Berlin.
RUDOLF LEUCKART, Giessen.
THEODOR MOMMSEN, Berlin.
ADOLPHE PICTET, Geneva.
CHRISTIAN FRIEDRICH SCHÖNBEIN, Basle.
KARL THEODOR VON SIEBOLD, Munich.

II. BRITISH.

JOHN STUART MILL, London.
GEORGE GABRIEL STOKES, Cambridge.
ALFRED TENNYSON, Freshwater, Isle of Wight.

The Council reported that they had awarded the Makdougall Brisbane Prize for the biennial period 1862–64 to JOHN DENIS MACDONALD, R.N., F.R.S., Surgeon to H.M.S. "Icarus," for his Zoological Papers published in the Transactions of the Society during the period.

It was moved by Lord NEAVES, and seconded by Dr CHRISTISON, That the cordial thanks of the Society be given to His Grace the DUKE of ARGYLL for his services during the term of his office as President. This motion was carried by acclamation.

It was moved by Lord NEAVES, and seconded by Dr CHRISTISON, That it is expedient that those Fellows who have filled the office of President should, on retiring from that office, become Honorary Vice-Presidents of the Society, and that it be remitted to the Council to carry this into effect. This motion was agreed to unanimously.

With the view of supplying an omission in the retirement of the late Treasurer, J. T. GIBSON-CRAIG, Esq., Dr CHRISTISON moved, seconded by Professor BALFOUR, that the cordial thanks of the Society be returned to Mr J. T. GIBSON-CRAIG, for his efficient services as Treasurer, and for the great attention which he had paid to the business of the Society. This was carried unanimously.

On the motion of Dr BURT the following Gentlemen were appointed to audit the Treasurer's accounts:—

WILLIAM CHAMBERS, Esq. JAMES CUNNINGHAM, Esq. Dr STEVENSON MACADAM.

The Meeting then adjourned.

(Signed) PHILIP KELLAND, *V.P.*

Monday, 27th November 1865.

At a Statutory General Meeting, Professor KELLAND, V.P., in the Chair, the Minutes of the Statutory Meeting of 28th November 1864 were read and confirmed.

The following Office-Bearers were elected for 1865–66.

Principal Sir DAVID BREWSTER, K.H., LL.D., D.C.L., President.
His Grace the DUKE of ARGYLL, Honorary Vice-President, having filled the Office of President.

Professor KELLAND,
Hon Lord NEAVES,
Principal FORBES,
Professor INNES,
Professor LYON PLAYFAIR, C.B.,
D. MILNE-HOME, Esq.
} Vice-Presidents.

Dr JOHN HUTTON BALFOUR, General Secretary.
Dr GEORGE JAMES ALLMAN, Professor P. GUTHRIE TAIT, } Secretaries to the Ordinary Meetings.
DAVID SMITH, Esq., Treasurer.
Dr DOUGLAS MACLAGAN, Curator of Library and Museum.

COUNCILLORS.

A. KEITH JOHNSTON, Esq.	Sir JAMES COXE, M.D.
Rev. Dr STEVENSON.	Rev. Dr BLAIKIE.
Dr STEVENSON MACADAM.	Dr CHRISTISON.
Hon. Lord JERVISWOODE.	Dr A. CRUM BROWN.
JAMES T. GIBSON-CRAIG, Esq.	Dr BURT.
EDWARD SANG, Esq.	Professor MACDOUGALL.

The following Gentlemen were proposed as Honorary Fellows of the Society:—

I. FOREIGN.

ANGELO SECCHI, Observatory, Rome.

II. BRITISH.

Lieut-General EDWARD SABINE, R.A., President of the Royal Society of London.
CHARLES DARWIN, Esq., M.A., Down, Bromley, Kent.
ARTHUR CAYLEY, Esq., Professor of Mathematics, Cambridge.

It was resolved that in future the accounts should be audited by a professional auditor, and that they should be presented at the general meeting in November.

It was moved by Dr BURT, and seconded by JOHN BLACKWOOD, Esq., that GEORGE AULDJO JAMIESON, Esq., should be appointed to audit the Treasurer's accounts. The motion was agreed to.

The Meeting adjourned.

(Signed) JAMES COXE, *Chairman.*

Monday, November 26, 1866.

At a Statutory General Meeting, Sir JAMES COXE, Councillor, in the Chair, the Minutes of the Statutory Meeting of 27th November 1865 were read and confirmed.

The following Office-Bearers were elected for 1866–67:—

Principal Sir DAVID BREWSTER, K.H., LL.D., D.C.L., President.

His GRACE THE DUKE OF ARGYLL, Honorary Vice-President, having filled the Office of President.

Hon. Lord NEAVES,
Principal FORBES,
Professor INNES,
Prof. LYON PLAYFAIR, C.B.,
D. MILNE-HOME, Esq.,
Dr CHRISTISON,
} Vice Presidents.

Dr JOHN HUTTON BALFOUR, General Secretary.

Dr GEORGE JAMES ALLMAN,
Professor TAIT,
} Secretaries to Ordinary Meetings.

DAVID SMITH, Esq., Treasurer.

Dr DOUGLAS MACLAGAN, Curator of Library and Museum.

COUNCILLORS.

JAMES T. GIBSON-CRAIG, Esq.	Professor KELLAND.
EDWARD SANG, Esq.	Dr MATTHEWS DUNCAN.
Sir JAMES COXE, M.D.	WILLIAM TURNER, M.B.
Rev. Dr BLAIKIE.	Dr JOHN MUIR.
Dr A. CRUM BROWN.	Rev. THOMAS BROWN.
Dr BURT.	JAMES SANDERSON, Esq.

The Treasurer's book was laid on the table, with the Auditor's Report.

On the motion of Dr BURT, seconded by Dr SELLER, GEORGE AULDJO JAMIESON, Esq., was elected Auditor for next year.

The Council recommended the election of the following Honorary Fellows to fill up the vacancies caused by the death of Professor ROGERS and of DR WHEWELL:—

I. FOREIGN.

MICHEL EUGENE CHEVREUL, Paris.

II. BRITISH.

THOMAS CARLYLE, Esq., London.

The SECRETARY stated that during last year the following vacancies had occurred in the List of the Society:—

Members Deceased,	. .	14
,, Resigned,	. .	4
,, Cancelled,	. .	4
	Total,	22

The present number of Ordinary Fellows was 276.

The Meeting adjourned.

LIST OF MEMBERS ELECTED.

January 3, 1865.

ALFRED R. CATTON, M.A. Rev. FRANCIS REDFORD, M.A.

January 16, 1865.

JAMES STEVENSON, Esq.

March 6, 1865.

Dr JOHN MOIR.

April 3, 1865.

JAMES POWRIE, Esq. CHARLES JENNER, Esq.

April 17, 1865.

CHARLES LAWSON, Jun., Esq.

December 4, 1865.

Dr ALEXANDER KEILLER.

December 18, 1865.

The Right Rev. Bishop MORRELL. WILLIAM EUING, Esq.

January 2, 1866.

Dr FRASER THOMSON. JOHN M'CULLOCH, Esq.
Dr T. GRAINGER STEWART. Colonel Sir JAMES E. ALEXANDER of Westerton.

January 15, 1866.

Dr CHARES MOREHEAD. Professor DAVID MASSON.
DAVID DOUGLAS, Esq.

February 5, 1866.

JOHN MACNAIR, Esq. Professor SPENCE.
THOMAS NELSON, Esq.

February 19, 1866.

ADAM BLACK, Esq. ALEXANDER MACDUFF, Esq. of Bonhard.
THOMAS CONSTABLE, Esq. Dr JAMES DUNSMURE.
Dr ARTHUR MITCHELL.

March 5, 1866.

Dr PATRICK HERON WATSON.

April 2, 1866.

Dr JOHN SMITH. JAMES FALSHAW, Esq., C.E.

April 16, 1866.

JOHN K. WATSON, Esq.
DAVID CHALMERS, Esq.

January 7, 1867.

T. B. JOHNSTON, Esq.
DAVID DAVIDSON, Esq.
Sir GEORGE HARVEY.
GEORGE F. BARBOUR, Esq. of Bonskeid.
PETER WADDELL, Esq.
GEORGE STIRLING HOME DRUMMOND, Esq. of Ardoch.
Professor FULLER.

January 21, 1867.

Dr ANDREW GRAHAM.
A. H. BRYCE, LL.D.
Dr ARTHUR GAMGEE.
WILLIAM TURNBULL, Esq.
FRANCIS DEAS, LL.B.
Sheriff HALLARD.

February 4, 1867.

Dr THOMAS R. FRASER.
THOMAS ANNANDALE, Esq.
Dr D. R. HALDANE.

February 18, 1867.

JOHN M. M'CANDLISH, Esq., W.S.

March 4, 1867.

JAMES DONALDSON, LL.D.
JAMES RICHARDSON, Esq.

March 18, 1867.

JAMES H. B. HALLEN, Esq.

April 1, 1867.

HENRY DIRCKS, Esq., C.E.

April 15, 1867.

Dr CHARLES GAINER.
WILLIAM KEDDIE, Esq.

April 29, 1867.

Rev. Dr LINDSAY ALEXANDER.

LIST OF THE PRESENT ORDINARY MEMBERS,

Corrected up to November 1, 1867.

IN THE ORDER OF THEIR ELECTION.

PRESIDENT.

PRINCIPAL SIR DAVID BREWSTER, K.H., LL.D., D.C.L.

HONORARY VICE-PRESIDENT, HAVING FILLED THE OFFICE OF PRESIDENT.

HIS GRACE THE DUKE OF ARGYLL, K.T.

Date of Election.

1808 James Wardrop, Esq., F.R.C.S.E., London.
Sir David Brewster, K.H., LL.D., F.R.S., Lond., *Principal of the University of Edinburgh.*
1812 Sir George Clerk, Bart., F.R.S., Lond.
1818 Patrick Miller, M.D., *Exeter.*
1820 Charles Babbage, F.R.S., Lond.
Sir John F. W. Herschel, Bart., F.R.S., Lond.
Dr William Macdonald, F.R.C.P.E., *Professor of Natural History, St Andrews.*
1821 Robert Hamilton, M.D., F.R.C.S.E.
1822 George A. Walker-Arnott, LL.D., *Professor of Botany, Glasgow.*
Sir James South, F.R.S., Lond.
Sir W. C. Trevelyan, Bart., *Wallington, Northumberland.*
1823 Captain Thomas David Stuart, *of the Hon. East India Company's Service.*
Warren Hastings Anderson, Esq.
Alexander Thomson, Esq., *of Banchory.*
Liscombe John Curtis, Esq., *Ingsdon-House, Devonshire.*
Robert Christison, M.D., *Professor of Materia Medica.*
1824 Robert E. Grant, M.D., *Professor of Comparative Anatomy, University College, London.*
Rev. Dr William Muir, *one of the Ministers of Edinburgh.*
1827 Very Rev. Edward Bannerman Ramsay, M.A. Camb., LL.D.
1828 John Forster, Esq., *Architect, Liverpool.*
Thomas Graham, M.A., D.C.L., F.R.S., *Master of the Mint, London.*
David Milne-Home, Esq., *Advocate, of Milne-Graden and Wedderburn*
Dr Manson, *Nottingham.*
1829 A. Colyar, Esq.
Right Hon. Sir William Gibson-Craig, Bart. *of Riccarton.*

Date of Election.	
1829	Right Honourable Lord Colonsay.
	Venerable Archdeacon Sinclair, *Kensington.*
	James Walker, Esq., W.S.
1830	J. T. Gibson-Craig, Esq., W.S.
	James Syme, Esq., *Professor of Clinical Surgery.*
	Thomas Barnes, M.D., *Carlisle.*
1831	James D. Forbes, D.C.L., F.R.S., Lond., *Principal of the United College, St Andrews.*
1832	Montgomery Robertson, M.D.
1833	Rear-Admiral Sir Alexander Milne, R.N.
	His Grace the Duke of Buccleuch, K.G., *Dalkeith Palace.*
	Alexander Hamilton, LL.B., W.S.
1834	Mungo Ponton, Esq., W.S., *Clifton, Bristol.*
	Isaac Wilson, M.D., F.R.S., Lond.
	Patrick Boyle Mure Macredie, Esq., *Advocate, Perceton.*
	William Sharpey, M.D., LL.D., F.R.S., *Professor of Anatomy, University College, London.*
1835	John Hutton Balfour, A.M., M.D., F.R.S., *Professor of Medicine and Botany.*
	William Brown, Esq., F.R.C.S.E.
	Robert Mayne, Esq.
1836	David Rhind, Esq., *Architect.*
1837	John Scott Russell, Esq., A.M., *London.*
	Archibald Smith, Esq., M.A., Camb., F.R.S., *Lincoln's Inn, London.*
	Richard Parnell, M.D.
	Peter D. Handyside, M.D., F.R.C.S.E.
1838	Thomas Mansfield, Esq., *Accountant.*
1839	David Smith, Esq., W.S.
	Adam Hunter, M.D., F.R.C.S.E.
	Rev. Philip Kelland, A.M., F.R.S., *Professor of Mathematics.*
	Francis Brown Douglas, Esq, *Advocate.*
1840	Alan A. Maconochie Welwood, Esq., *of Meadowbank and Pitliver.*
	Martyn J. Roberts, Esq., *Fort-William.*
	Robert Chambers, LL.D.
	Sir John M'Neill, G.C.B., LL.D.
	Sir William Scott, Bart., *of Ancrum.*
	Right Rev. Bishop Terrot.
	Edward J. Jackson, Esq.
	John Mackenzie, Esq.
	James Anstruther, Esq., W.S.
1841	John Miller, Esq., *of Leithen.*
	James Dalmahoy, Esq.
1842	James Thomson, Esq., *Civil-Engineer, London.*
	John Davy, M.D., *Inspector-General of Army Hospitals.*
	Robert Nasmyth, Esq., F.R.C.S.E.
1843	A. D. Maclagan, M.D., *Professor of Medical Jurisprudence.*
	John Rose Cormack, M.D., F.R.C.P.E., *Orleans, France.*

Date of Election.

1843 Allen Thomson, M.D., F.R.S., *Professor of Anatomy, Glasgow.*
Joseph Mitchell, Esq., *Civil Engineer Inverness.*
Andrew Coventry, Esq., *Advocate.*
John Hughes Bennett, M.D., *Professor of Physiology.*
D. Balfour, Esq., *of Trenaby.*
Henry Stephens, Esq.
1844 Archibald Campbell Swinton, Esq., *of Kimmerghame.*
James Begbie, M.D., F.R.C.P.E.
Sir James Y. Simpson, Bart., M.D., *Professor of Midwifery.*
David Stevenson, Esq., *Civil Engineer.*
Thomas R. Colledge, M.D., F.R.C.P.E.
1845 John G. M. Burt, M.D., F.R.C.P.E.
Thomas Anderson, M.D., *Professor of Chemistry, Glasgow.*
1846 A. Taylor, M.D., *Pau.*
Alexander J. Adie, Esq., *Civil Engineer.*
L. D. B. Gordon, Esq., C.E.
L. Schmitz, LL.D., Ph.D., *International Institution, London.*
Charles Piazzi Smyth, Esq., F.R.S., *Professor of Practical Astronomy.*
1847 Sir William Thomson, M.A., Camb. LL.D., F.R.S., *Professor of Natural Philosophy, Glasgow.*
J. H. Burton, Esq., LL.D., *Advocate.*
James Nicol, Esq., *Professor of Natural History, Aberdeen.*
William Macdonald Macdonald, Esq., *of St Martins.*
John Wilson, Esq., *Professor of Agriculture.*
Moses Steven, Esq., *of Bellahouston.*
1848 Thomas Stevenson, Esq., C.E.
James Allan, M.D., *Inspector of Hospitals, Portsmouth.*
Henry Davidson, Esq.
William Swan, Esq., *Professor of Natural Philosophy, St Andrews.*
Patrick James Stirling, Esq.
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ERRATA.

In Theorem 7, page 41, and in Theorem VII., page 44, *for* $\left\{\begin{array}{l} A_1B,\ BC,\ \&c. \\ B_1C,\ CA,\ \&c. \\ C_1A,\ AB,\ \&c. \end{array}\right\}$ *read* $\left\{\begin{array}{l} AB_1,\ BC,\ \&c. \\ BC_1,\ CA,\ \&c. \\ CA_1,\ AB,\ \&c. \end{array}\right.$

In Theorem LXXII. page 52, the sentence ought to end at the *comma;* the rest is obviously erroneous.

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